

System Identification

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Transient Analysis of Step and Impulse Responses

Sometimes a simple first or second-order model is sufficient; transient analysis offers an easy way to obtain it.

Closest relation to prior knowledge from system theory $\Rightarrow$ gentle transition towards other techniques.

Classification (continued)

Step and impulse response models can be seen as **nonparametric** models, for those steps in which we study the graph of the response.

However, based on information from the graph, we will in the end find a transfer function – a **parametric** model.

These models are best classified as **gray-box**.

The study of these models is called *transient analysis*, since it relies in a large part on the transient regime of the response.

- 1 Recap: Continuous-time linear models
- 2 Step response models of first-order systems
- 3 Step response models of second-order systems
- 4 Impulse response models of first-order systems
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Homogeneity: If for input $u(t)$ the system responds with output $y(t)$; then for input $\alpha u(t)$ the system will respond with $\alpha y(t)$.

- s is called *complex* argument (it is a complex number), and the Laplace transform can be seen as taking a function from the time domain t to the complex domain s .
- The motivation is that many signal operations common in engineering (differentiation, integration, etc.) become much simpler in the s domain.

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First order system: Motivating example

First-order systems are common. Typical example: a thermal system. Consider an object at temperature θ_1 (output variable) placed in an environment at temperature θ_2 (input variable). Then:

$$C\dot{\theta}_1(t) = \frac{\theta_2(t) - \theta_1(t)}{R}$$

where C is the thermal capacitance and R is the thermal resistance.

Applying the Laplace transform on both sides:

$$Cs\Theta_1(s) = \frac{\Theta_2(s) - \Theta_1(s)}{R}$$

leading to the transfer function:

$$H(s) = \frac{\Theta_1(s)}{\Theta_2(s)} = \frac{1}{CRs + 1}$$

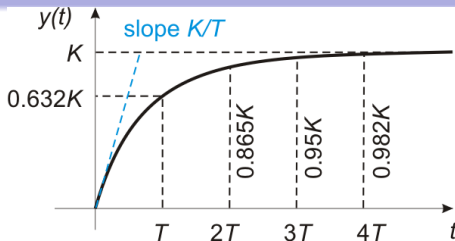
- K is the gain ($= 1$ in the example)
- T is the time constant ($= CR$ in the example)

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Ideal step input



$$u_S(t) = \begin{cases} 0 & t \leq 0 \\ 1 & t > 0 \end{cases}$$



Solving the differential equation for $y(t)$ (or easier: solve for $Y(s)$ and then apply inverse Laplace transform $\mathcal{L}^{-1}$), we get:

$$y(t) = K(1 - e^{-t/T})$$

from where:

$$\lim_{t \rightarrow \infty} y(t) = K(1 - 0) = K$$

$$\dot{y}(t) = \frac{K}{T} e^{-t/T}, \quad \dot{y}(0) = \frac{K}{T} e^0 = \frac{K}{T}$$

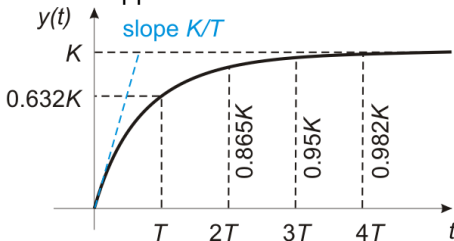
$$y(T) = K(1 - e^{-1}) \approx 0.632K$$

and similar for $t = 2T, 3T, 4T$ (see figure).

Determining the parameters

So far, everything known from:  Sys. Theory,  Process Modeling.

Now, consider we are given a step response of an unknown system. We can use it to find an approximate transfer function of the system.



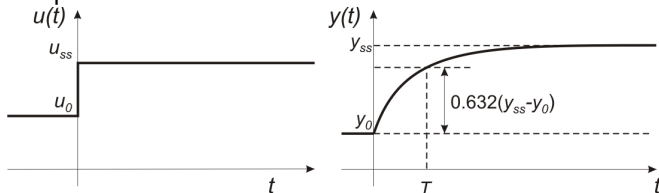
Algorithm for **system identification**

- 1 Read the steady-state value. That is the gain K .
- 2 Determine the time value where the output reaches 0.632 of its steady state value. That is the time constant T .

Nonzero initial conditions

In practice, we often cannot use ideal step signals: the system must be kept around a safe/profitable operating point. In particular, assume the system was in steady-state at y_0 with input held constant at u_0 .

Real-life step inputs are often rectangular signals as below. The system response is therefore nonstandard.

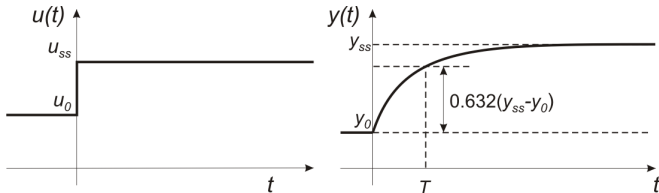


But recall linearity properties. Our new input is

$u(t) = u_0 + (u_{ss} - u_0)u_S(t)$ with $u_S(t)$ the ideal step input. Then, denoting the ideal step response by $y_S(t)$, we have the new output:

$$y(t) = y_0 + (u_{ss} - u_0)y_S(t)$$

simply a shifted and scaled version.



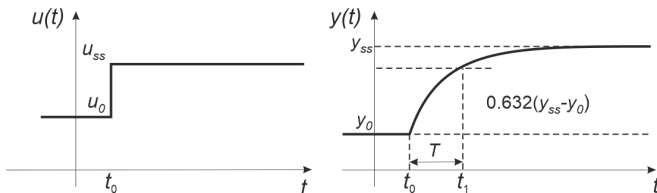
Then we obtain:

$$y_{ss} = y_0 + (u_{ss} - u_0)K$$

$$y(T) = y_0 + 0.632(y_{ss} - y_0)$$

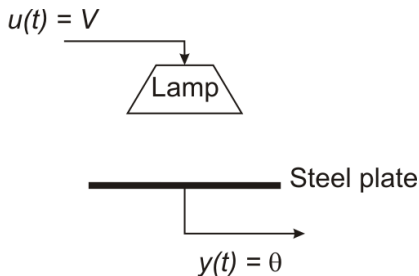
Nonzero step time

The start time of the step may also be different from 0, solved easily by shifting the time axis. Such a situation can occur for any of the step and impulse responses throughout the remainder of this part, and it is always handled the same way. We will provide details for the step responses, and the same idea can be applied for impulse responses.



- 1 Read u_0, y_0, u_{ss}, y_{ss} the initial and steady-state values of the input and output signals. Compute $K = \frac{y_{ss} - y_0}{u_{ss} - u_0}$.
- 2 Read t_0 the start time of the step, t_1 the time where the output raises 0.632 of the difference. Compute $T = t_1 - t_0$.

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Consider the thermal system in the figure (different from the example above). The input is the voltage V applied to the lamp, the output is the temperature θ read by a thermocouple at the back of the steel plate.

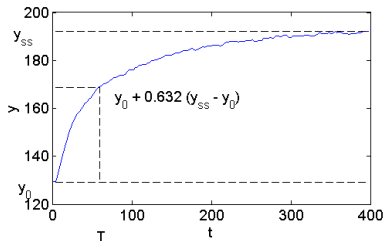
We use the first step for identification, and the others for validation.

Sampling period

Definition: The sampling period T_s is the continuous time-interval between two successive discrete-time sampling points (of the input, output, or other signals in the system).

Don't confuse sampling period T_s with the time constant T multiplied by complex argument s !

Thermal system: Model and parameters



We have $y_{ss} \approx 192^\circ \text{C}$, $y_0 \approx 129^\circ \text{C}$. Also, the input $u_{ss} = 9 \text{ V}$ and (from the experiment) we know that $u_0 = 6 \text{ V}$. Therefore:

$$K = \frac{y_{ss} - y_0}{u_{ss} - u_0} \approx \frac{192 - 129}{9 - 6} \approx 21$$

Further, $y(T) = y_0 + 0.632(y_{ss} - y_0) \approx 169$, and identifying this point on the graph we get $T \approx 60$.

Thermal system: Transfer function model

$$\hat{K} = 21$$

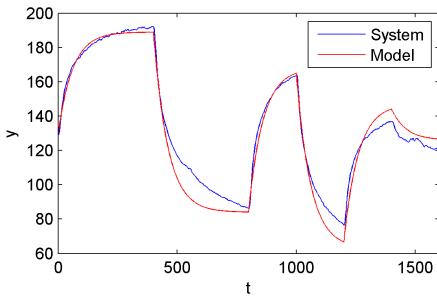
$$\hat{T} = 60$$

$$\hat{H}(s) = \frac{\hat{K}}{\hat{T}s + 1} = \frac{21}{60s + 1}$$

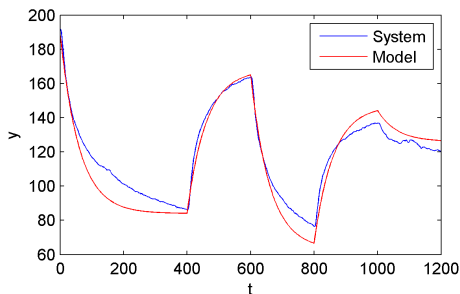
The “hat” notation makes explicit the fact that the model is an approximation.

Matlab: $H = \text{tf}(\text{num}, \text{den})$, with polynomials represented as vectors of coefficients in decreasing powers of s .

(Note: Actual calculations done in double representation with Matlab, so using the numbers given in the slides will lead to slightly different results. This remark applies to all the examples.)



Thermal system: Validation (continued)



Mean squared error (MSE) on the validation data (second and further steps):

$$J = \frac{1}{N} \sum_{k=1}^N e^2(k) = \frac{1}{N} \sum_{k=1}^N (\hat{y}(k) - y(k))^2 \approx 62.10$$

Recall that the data is actually sampled in discrete time, so a meaningful MSE can be computed.

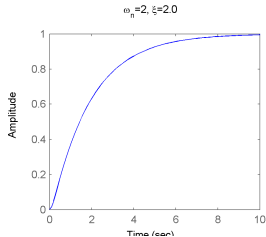
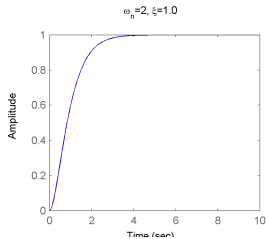
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$$H(s) = \frac{K\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

where:

- K is the gain ($= \frac{1}{k}$ in the example)
- ξ is the damping ($= 0$ in the example)
- ω_n is the natural frequency ($= \sqrt{k/m}$ in the example)

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Underdamped 2nd order step response

We are mostly interested in the underdamped case ($\xi \in (0, 1)$)



Solving for $y(t)$ we get:

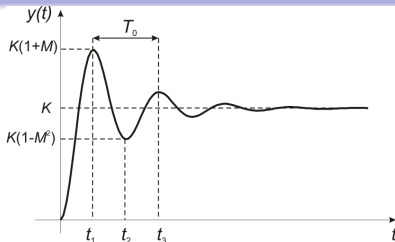
$$y(t) = K \left[1 - \frac{1}{\sqrt{1 - \xi^2}} e^{-\xi \omega_n t} \sin(\omega_n \sqrt{1 - \xi^2} t + \arccos \xi) \right]$$

The graph shows the time response $y(t)$ of a second-order system. The vertical axis is labeled $y(t)$ and the horizontal axis is labeled t . The curve starts at the origin, rises to a peak at t_1 with value $K(1+M)$, crosses the steady-state value K at t_2 , reaches a minimum at t_3 with value $K(1-M^2)$, and then oscillates with decreasing amplitude around K . The period T_0 is marked between t_1 and t_3 .

$$y(t_m) = K[1 + (-1)^{m+1} M^m], \text{ where overshoot } M = e^{-\frac{\xi \pi}{\sqrt{1-\xi^2}}}$$

The graph shows a function $y(t)$ plotted against time t . The curve starts at the origin $(0,0)$, rises to a maximum value, then oscillates with decreasing amplitude, eventually settling to a constant value.

Determining the parameters



Algorithm

- 1 Determine steady-state output value y_{ss} . That is the gain K .
- 2 Determine overshoot M , (a) from the first peak: $M = \frac{y(t_1) - y_{ss}}{y_{ss}}$, or
(b) from ratio of first valley and first peak: $M = \frac{y_{ss} - y(t_2)}{y(t_1) - y_{ss}}$.
- 3 Solve $M = e^{-\frac{\xi \pi}{\sqrt{1 - \xi^2}}}$, leading to $\xi = \frac{\log 1/M}{\sqrt{\pi^2 + \log^2 1/M}}$
- 4 Read oscillation period as the time between first two peaks

$$T_0 = t_3 - t_1 = \frac{2\pi}{\omega_n \sqrt{1-\xi^2}}; \text{ or } T_0 = 2(t_2 - t_1). \text{ Then,}$$

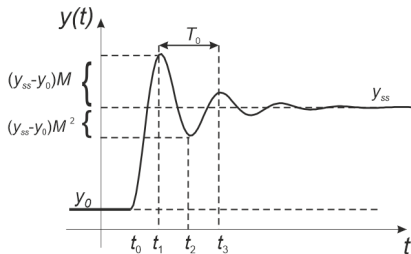
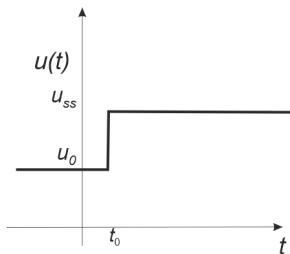
$$\omega_n = \frac{2\pi}{T_0 \sqrt{1-\xi^2}}, \text{ or } \omega_n = \frac{2}{T_0} \sqrt{\pi^2 + \log^2 1/M}.$$

1 Gain $K = \frac{y_{ss} - y_0}{u_{ss} - u_0}$.

2 Overshoot (a) $M = \frac{y(t_1) - y_{ss}}{y_{ss} - y_0}$ (we need to subtract y_0), or (b)

$$M = \frac{y_{ss} - y(t_2)}{y(t_1) - y_{ss}} \text{ (no change in this formula).}$$

ξ, T_0 : same as before.

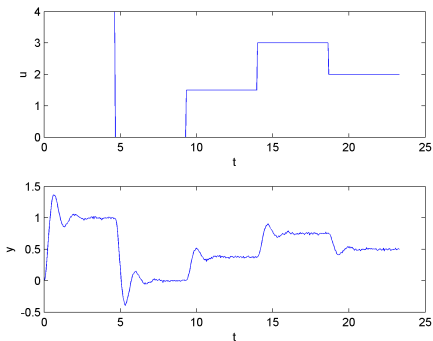


Like in the first-order case, we simply shift everything by the starting time t_0 of the step. This has no impact in the algorithm as we use relative times to compute the oscillation period, anyway.

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2nd order example

Simulation example, 500 samples with sampling time ≈ 0.047 .



Note again the measurement noise. Also, while the experiment has zero initial condition ($u_0 = y_0 = 0$), the steps still have nonstandard values (different from 1).

We will use step 1 for identification, steps 3-5 for validation (using the fact that step 2 returns the system to zero initial condition).

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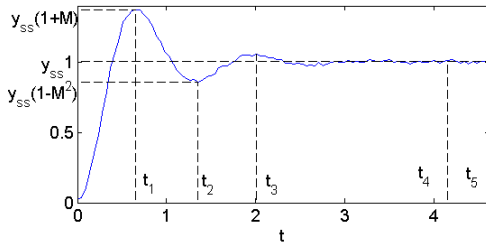


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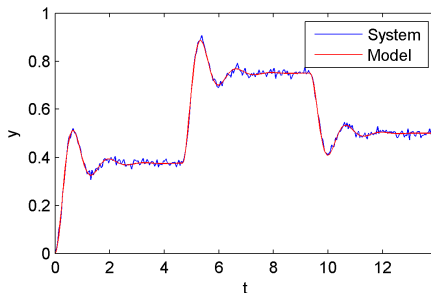
1. **Introduction**



- 1 Gain $K = \frac{y_{ss}-y_0}{u_{ss}-u_0} = \frac{y_{ss}}{u_{ss}} \approx 0.25$.
- 2 Overshoot $M = \frac{y(t_1)-y_{ss}}{y_{ss}-y_0} = \frac{y(t_1)-y_{ss}}{y_{ss}} \approx 0.36$.
- 3 Damping $\xi = \frac{\log 1/M}{\sqrt{\pi^2 + \log^2 1/M}} \approx 0.31$.
- 4 Period $T_0 = t_3 - t_1 \approx 1.31$, so natural frequency $\omega_n = \frac{2\pi}{T_0 \sqrt{1-\xi^2}} \approx 5.05$.

$$\hat{H}(s) = \frac{\hat{K}\hat{\omega}_n^2}{s^2 + 2\xi\hat{\omega}_n s + \hat{\omega}_n^2} = \frac{6.38}{s^2 + 3.09s + 25.51}$$

Example: Validation of transfer function model



Very good fit (not surprising since this is synthetic data).

Mean squared error (MSE):

$$J = \frac{1}{N} \sum_{k=1}^N e^2(k) = \frac{1}{N} \sum_{k=1}^N (\hat{y}(k) - y(k))^2 \approx 9.66 \cdot 10^{-5}$$

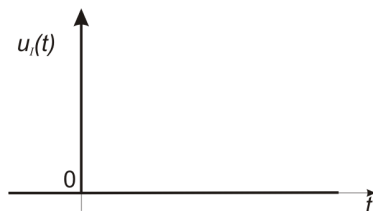
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Even when it is overdamped or critically-damped, at $t = 0$ a 2nd order system response will have a derivative of 0: it will be **tangent to the time axis**. In contrast, the tangent slope is K/T for 1st order systems.

- Gain K : difference between output levels / difference between input levels.
- 1st order: time constant T found on *time* axis when the *output* axis reaches 63.2% of the difference.
- 2nd order: period T_0 read on the graph, overshoot M computed from the peaks and valleys. ξ, ω_n follow.
- Graph inspection + mean squared error used to validate models.
- Averaging to reject noise in initial/steady-state values.
- Nonzero initial time and time delays handled by shifting the time values appropriately; time delay goes in transfer function.

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Ideal impulse input



The ideal impulse is the Dirac delta. An informal definition:

$$u_I(t) = \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases}$$

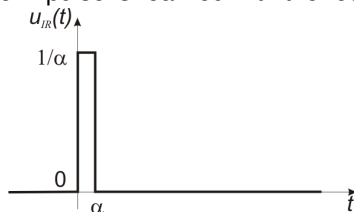
with the additional condition $\int_{-\infty}^{\infty} u_I(t) dt = 1$.

(In fact, the ideal impulse is not a function and requires the notion of distributions to be formally defined.)

Practical impulse realization

In practice, we cannot create signals of infinite amplitude.

So, an approximate impulse is realized with the rectangular signal:



$$u_{\text{IR}}(t) = \begin{cases} \frac{1}{\alpha} & t \in [0, \alpha) \\ 0 & \text{otherwise} \end{cases}$$

where $\alpha \ll$ (much smaller than the time constants in the system).

Note the signal still obeys $\int_{-\infty}^{\infty} u_{\text{IR}}(t)dt = 1$ (rectangle has area 1).

This approximate impulse will introduce differences (error) from the true impulse response, but for small α the error is not large. We develop the analysis in the ideal case, while the examples use the practical realization.

Useful property of impulse response

In the Laplace domain:

step input $U_S(s) = \frac{1}{s}$, impulse input $U_I(s) = 1$

Recall that the time-domain response of a system can be expressed as: $y(t) = \mathcal{L}^{-1}\{Y(s)\}$, and $Y(s) = H(s)U(s)$. So:

$$\begin{aligned} Y_S(s) &= \frac{1}{s} Y_I(s), & Y_I(s) &= s Y_S(s) \\ y_S(t) &= \int_0^t y_I(\tau) d\tau, & y_I(t) &= \dot{y}_S(t) \end{aligned}$$

The impulse response is the *derivative of the step response*.

5 Impulse response models of second-order systems

$$H(s) = \frac{K}{Ts + 1}$$

where:

- K is the gain
- T is the time constant

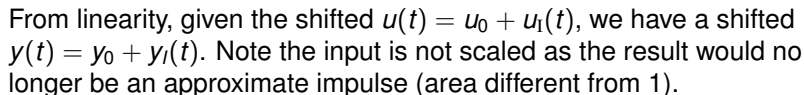
A graph showing the response $y(t)$ versus time t . The curve starts at K/T on the y -axis and decays exponentially. A dashed line indicates that at time T , the response is $0.368K/T$.

$$y_1(t) = \frac{K}{T} e^{-t/T}, \quad t \geq 0$$

from where:
$$\begin{cases} y_I(0) = \frac{K}{T} = y_{\max} \\ y_I(T) = \frac{K}{T} e^{-1} = y_{\max} e^{-1} \approx 0.368 y_{\max} \end{cases}$$

Note: $y_1(4T) = 0.0183y_{\max}$, so like for the step response, the output is roughly in steady state after $4T$.

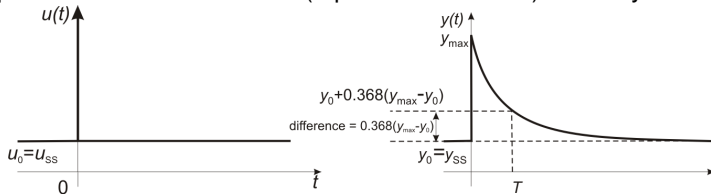
When the initial conditions are nonzero, the impulse is shifted along the vertical axis. Assume the system was in steady-state at y_0 with input held constant at u_0 .



Note $u_0 = u_{ss}, y_0 = y_{ss}$.

Determining the parameters

Consider now that we are given the impulse response of an unknown system. As in the step case, we can use this response to find an approximate transfer function (a parametric model) of the system.



We consider first nonzero initial conditions because that is actually a favorable situation: we have a reliable way to find the gain K .

Algorithm

- 1 Read the steady-state (or initial) output $y_{ss} = y_0$ and input $u_{ss} = u_0$. Then, $K = y_{ss}/u_{ss}$.
- 2 Read $y_{\max}$ and read the time constant T at the moment where the output decreases to 0.368 of the difference $y_{\max} - y_0$.

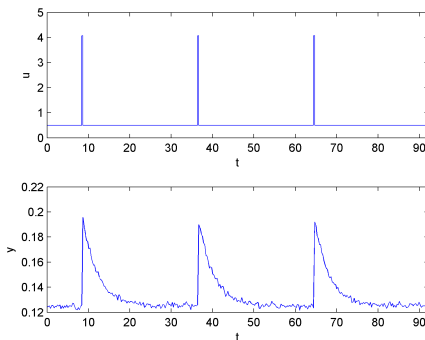
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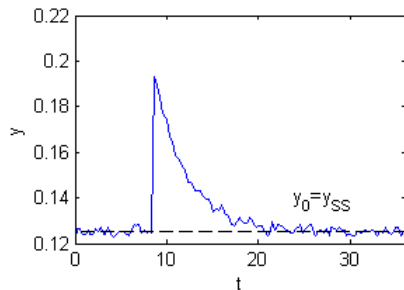
1st order example

Simulation example, 330 samples with sampling time $T_s = 0.28$ (30 samples are the initial steady-state regime, then 100 for each impulse response). The practical impulses are realized with $\alpha = T_s = 0.28$, amplitude $1/\alpha \approx 3.57$.



Note the measurement noise and the nonzero initial condition.

We will use impulse 1 for identification, impulses 2-3 for validation.

[illegible]

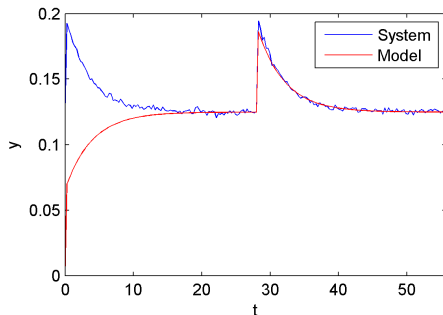
$$y_{ss} = y_0 \approx \frac{1}{11} \sum_{k=120}^{130} y(k) \approx 0.13$$

Example: Model and parameters (continued)

$$\hat{H}(s) = \frac{\hat{K}}{\hat{T}s + 1} = \frac{0.25}{3.92s + 1}$$

Example: Validation of transfer function model

Comparison of system data and model response for the validation data (second and third impulse responses):



The simulation does not take into account the nonzero initial condition of the system, hence the first part has large differences.

We will present a method to take into account initial conditions, which works not only for impulse signals, but for *any* input (step, etc.).

State space model of an n th order system

A (continuous-time) **state space model** of a linear system is a representation of the system in the following form:

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t)$$

where:

- x state vector, $x \in \mathbb{R}^n$ with n the order of the system
- u and y are the usual input and output. They can be vectors if the system has several inputs or outputs, but for us here, a scalar input and output are enough.
- A state matrix, B input matrix, C output matrix, and D feedthrough matrix. They have appropriate dimensions: $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times 1}$ (a vector, due to scalar input), $C \in \mathbb{R}^{1 \times n}$ (a vector, due to scalar output), $D \in \mathbb{R}$ (a scalar, usually 0).

State space model of a general 1st order system

Starting from transfer function model:

$$H(s) = \frac{K}{Ts + 1} = \frac{Y(s)}{U(s)}$$

and moving back to the time domain we get:

$$\dot{y}(t) = \frac{-1}{T}y(t) + \frac{K}{T}u(t)$$

By simply taking $x = y$ (recall that the system has order 1 so a single state suffices), we can write:

$$\begin{aligned}\dot{x}(t) &= -\frac{1}{T}x(t) + \frac{K}{T}u(t) \\ y(t) &= x(t)\end{aligned}$$

so our state space model has $A = -\frac{1}{T}$, $B = \frac{K}{T}$, $C = 1$, $D = 0$.

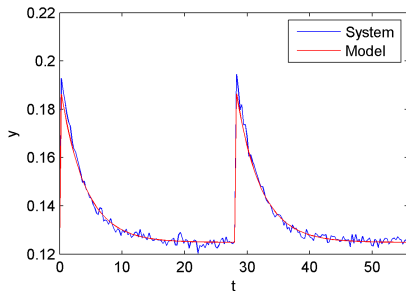
Back to example: (Approximate) state space model

$$\begin{aligned}\dot{x}(t) &= -\frac{1}{\hat{T}}x(t) + \frac{\hat{K}}{\hat{T}}u(t) = -0.26x(t) + 0.06u(t) \\ y(t) &= x(t)\end{aligned}$$

Matlab: $H_{SS} = SS(A, B, C, D)$

Example: Validation with correct initial condition

To take the initial condition into account, we simply set $x(0) = y_0$ when starting the simulation.

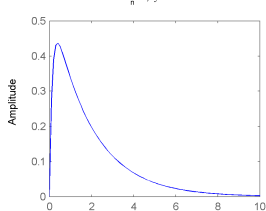
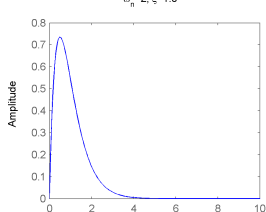


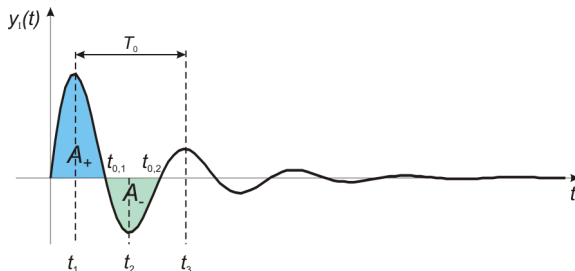
Mean squared error (MSE) on the validation data:

$$J = \frac{1}{N} \sum_{k=1}^N e^2(k) \approx 3.74 \cdot 10^{-6}$$

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- K is the gain
- ξ is the damping factor
- ω_n is the natural frequency



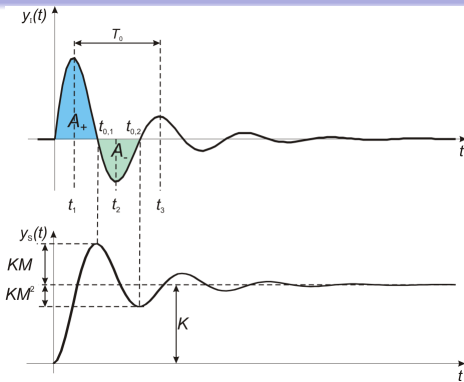


Using the derivative of the step response we already computed, we have the impulse response:

$$y_I(t) = \frac{K\omega_n}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t} \sin(\omega_n \sqrt{1-\xi^2} t)$$

Already note that the oscillation period is unchanged, so $T_0 = t_3 - t_1 = 2(t_2 - t_1)$.

Underdamped impulse response: areas



Since $y_S(t) = \int_0^t y_I(\tau) d\tau$, and remembering the sizes of the first peak and valley in the step response as a function of the overshoot M :

$$A_+ = \int_0^{t_{0,1}} y_I(\tau) d\tau = y_S(t_{0,1}) = K + KM, \quad A_- = - \int_{t_{0,1}}^{t_{0,2}} y_I(\tau) d\tau =$$

$$= -[y_S(t_{0,2}) - y_S(t_{0,1})] = -[K - KM^2 - (K + KM)] = KM^2 + KM$$

$$\frac{A_-}{A_+} = \frac{KM^2 + KM}{K + KM} = M$$

From the steady-state values we can estimate the **gain**: $K = \frac{y_{ss}}{u_{ss}}$.
There is no change in T_0 , but the areas must now be found **relative to the steady-state value**:

$$A_+ = \int_0^{t_{0,1}} (y(\tau) - y_0) d\tau = K + KM$$

$$A_- = - \int_{t_{0,1}}^{t_{0,2}} (y(\tau) - y_0) d\tau = \int_{t_{0,1}}^{t_{0,2}} (y_0 - y(\tau)) d\tau = KM^2 + KM$$

$$A_+ = \int_0^{t_{0,1}} (y(t) - y_0) dt \approx T_s \sum_{k=0}^{k_{0,1}} (y(k) - y_0)$$

$$A_- = \int_{t_{0,1}}^{t_{0,2}} (y_0 - y(t)) dt \approx T_s \sum_{k=k_{0,1}}^{k_{0,2}} (y_0 - y(k))$$

with $k_{0,1}, k_{0,2}$ discrete-time (sample) indices corresponding to $t_{0,1}, t_{0,2}$.

Determining the parameters

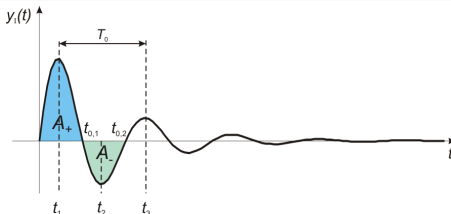
Given the impulse response of an unknown system, the transfer function is found as follows:

Algorithm

- 1 Read steady-state output y_{ss} , and u_{ss} . The gain is $K = \frac{y_{ss}}{u_{ss}}$.
- 2 Read time values where $y(t)$ crosses y_{ss} : $t_{0,1}$, $t_{0,2}$. Compute areas $A_+ = \int_{t_{0,1}}^{t_{0,2}} (y(\tau) - y_{ss}) d\tau$, $A_- = \int_{t_{0,1}}^{t_{0,2}} (y_{ss} - y(\tau)) d\tau$. Find overshoot $M = \frac{A_-}{A_+}$.
- 3 The damping factor is $\xi = \frac{\log 1/M}{\sqrt{\pi^2 + \log^2 1/M}}$.
- 4 Read time values at peaks, t_1 , t_3 (or peak and valley t_1 , t_2). Find the oscillation period $T_0 = t_3 - t_1$, or $T_0 = 2(t_2 - t_1)$.
- 5 Natural frequency $\omega_n = \frac{2\pi}{T_0\sqrt{1-\xi^2}}$, or $\omega_n = \frac{2}{T_0} \sqrt{\pi^2 + \log^2 1/M}$.

Note the relationships between M , T_0 , ξ , and ω_n are true regardless of the response type, so algorithm steps 3 and 5 use the same formulas as in the step-response case.

Determining the gain in zero initial conditions



We solve $\dot{y}(t) = 0$ to get t_1 for the first peak, and replace it in $y(t)$ to get the value at the peak. After some calculation we obtain:

$$y(t_1) = K\omega_n e^{-\frac{\xi \arccos \xi}{\sqrt{1-\xi^2}}}$$

which can be used to estimate the gain as $K = \frac{y(t_1)}{\frac{\xi \arccos \xi}{\omega_{ne} \sqrt{1-\xi^2}}}$. This

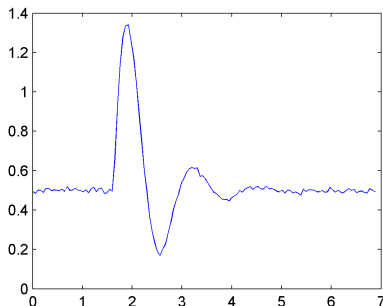
requires ξ and ω_n to be computed by the methods above, which can be done regardless of the initial condition.

For the same reasons as in the first-order case, this method is less accurate than determining the gain from nonzero steady-state values.

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Example: Steady-state values and gain

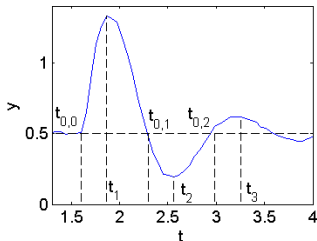


We use the graph to estimate a transfer function. We have

$$u_0 = u_{ss} = 2.$$

We determine the steady state output (equal to the initial output) by averaging the last 11 samples:

$$y_{ss} = y_0 \approx \frac{1}{11} \sum_{k=120}^{130} y(k) \approx 0.5$$



We read $t_{0,0} \approx 1.6$, $t_{0,1} \approx 2.3$, $t_{0,3} \approx 2.99$. Note the impulse is applied at time $t_{0,0} \neq 0$, so we need to take this into account.

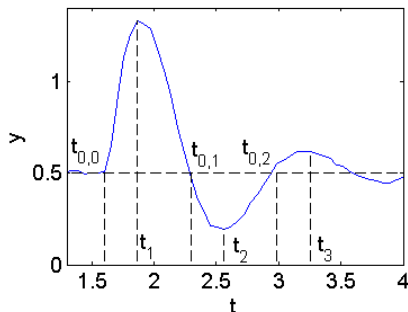
Recall area estimation:

$$A_+ = \int_{t_{0,0}}^{t_{0,1}} (y(t) - y_0) dt \approx T_s \sum_{k=k_{0,0}}^{k_{0,1}} (y(k) - y_0) \approx 0.34$$

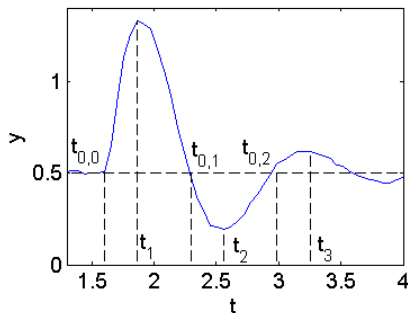
$$A_- = \int_{t_{0,1}}^{t_{0,2}} (y_0 - y(t)) dt \approx T_s \sum_{k=k_{0,1}}^{k_{0,2}} (y_0 - y(k)) \approx 0.12$$

with $k_{0,0}, k_{0,1}, k_{0,2}$ sample indices corresponding to $t_{0,0}, t_{0,1}, t_{0,2}$.

Example: Damping factor (continued)



From these areas, $M = \frac{A_-}{A_+} \approx 0.36$, and $\xi = \frac{\log 1/M}{\sqrt{\pi^2 + \log^2 1/M}} \approx 0.31$.



We read $t_1 \approx 1.92$ and $t_3 \approx 3.2$, leading to $T_0 = 1.28$. From this,

$$\omega_n = \frac{2\pi}{T_0 \sqrt{1-\xi^2}} \approx 5.16.$$

$$\hat{\xi} = 0.31$$

$$\hat{H}(s) = \frac{\hat{K}\hat{\omega}_n^2}{s^2 + 2\xi\hat{\omega}_n s + \hat{\omega}_n^2} = \frac{6.64}{s^2 + 3.21s + 26.68}$$

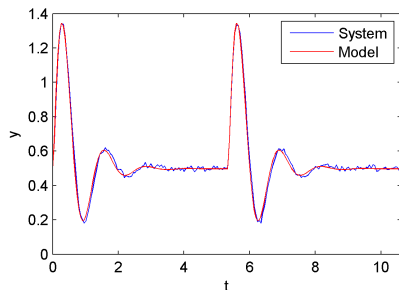
$$\begin{aligned} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} &= \begin{bmatrix} x_2(t) \\ -2\xi\omega_n x_2(t) - \omega_n^2 x_1(t) + K\omega_n^2 u(t) \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\xi\omega_n \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ K\omega_n^2 \end{bmatrix} u(t) \\ y(t) = x_1(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + 0u(t) \end{aligned}$$

$$\begin{aligned}\dot{x}(t) &= \begin{bmatrix} 0 & 1 \\ -26.68 & -3.22 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 6.64 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) + 0u(t)\end{aligned}$$

where x is now the whole state vector, $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$.

Example: Validation

To take the initial condition into account, we set $x_1(0) = y_0$, $x_2(0) = 0$ when starting the simulation (we start from steady state, so $x_2(0) = \dot{y}(0) = 0$).



Mean squared error (MSE) on the validation data:

$$J = \frac{1}{N} \sum_{k=1}^N e^2(k) \approx 8 \cdot 10^{-4}$$

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$$H(s) = \frac{K}{Ts+1} e^{-sT}, \quad H(s) = \frac{K\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} e^{-sT}$$

- Impulse response = derivative of the step response.
- Gain K : output/input in nonzero initial conditions, otherwise from maximum value.
- 1st order: time constant T found on *time* axis when the *output* axis reaches 36.8% of the difference.
- 2nd order: period T_0 read on the graph, overshoot M computed via numerical integration. ξ, ω_n follow.
- State-space model to handle nonzero initial conditions.