System Identification

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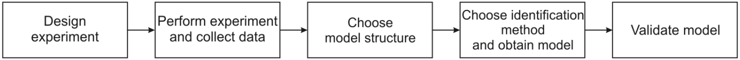
Part VI

Input signals

Motivation



Choosing inputs is the core of experiment design

All identification methods require inputs to satisfy certain conditions, for example:

- Transient analysis requires step or impulse inputs
- Correlation analysis preferably works with white-noise input
- ARX requires “sufficiently informative” inputs

Plan

In this part we:

- Revisit some types of input signals that were already used
- Describe a few new types of input signals
- Discuss choices and properties of input signals important for system identification
- Characterize the signals discussed using the properties introduced

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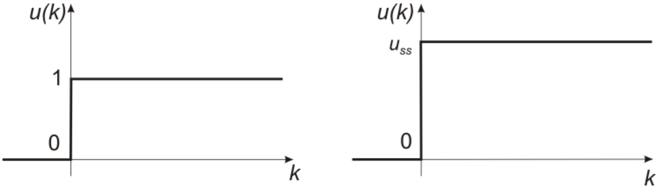
Step input



Left: Unit step:

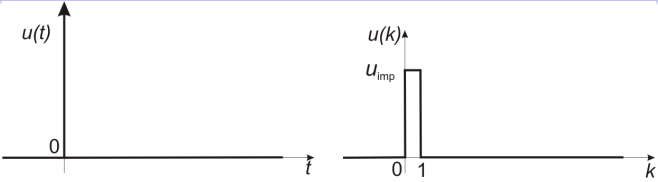
$$u(k) = \begin{cases} 0 & k < 0 \\ 1 & k \geq 0 \end{cases}$$

Right: Step of arbitrary magnitude:

$$u(k) = \begin{cases} 0 & k < 0 \\ u_{ss} & k \geq 0 \end{cases}$$

Remark: These are discrete-time reformulations of the continuous-time step variants.

Impulse input



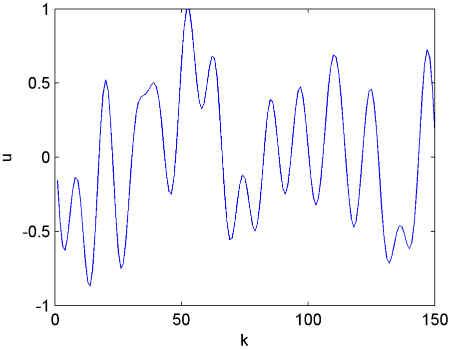
Recall that in discrete time, we cannot freely approximate the ideal impulse (left), since the signal can only change values at the sampling instants.

Right: Discrete-time impulse realization:

$$u(k) = \begin{cases} u_{\mathrm{imp}} & k = 0 \\ 0 & \text{otherwise} \end{cases}$$

- When $u_{\mathrm{imp}} = \frac{1}{T_s}$, the integral of the signal is 1 and we get an approximation of the continuous-time impulse.
- When $u_{\mathrm{imp}} = 1$ (e.g. in correlation analysis), we get a “unit” discrete-time impulse.

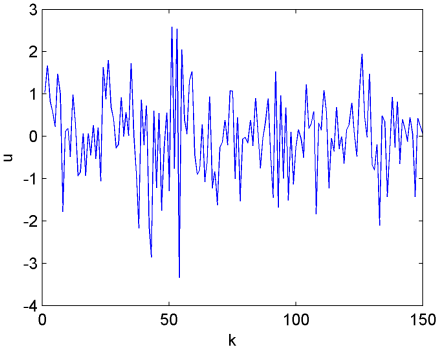
Sum of sines



$$u(k) = \sum_{j=1}^{m} a_j sin(\omega_j k + \varphi_j)$$

- a_j: amplitudes of the m component sines
- ω_j: frequencies, $0 \leq \omega_1 < \omega_2 < \ldots < \omega_m \leq \pi$
- φ_j: phases

White noise



Recall zero-mean white noise: mean 0, different steps uncorrelated.

In the figure, values were independently drawn from a zero-mean Gaussian distribution.

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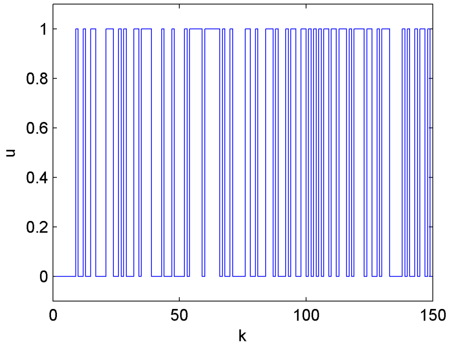
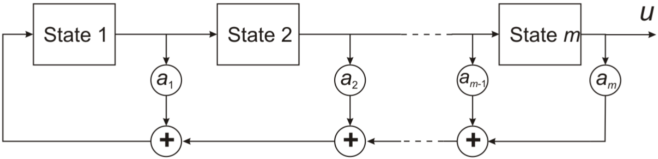
Pseudo-random binary sequence (PRBS)



A signal that switches between two discrete values, generated with a specific algorithm.

Interesting because it approximates white noise, and so it inherits some of the useful properties of white noise (formalized later).

PRBS generator



PRBS can be generated with a linear shift feedback register as in the figure. All signals and coefficients are binary (the states are bits).

At each discrete step $k \geq 0$:

- State x_i transfers to state x_{i+1}.
- State x_1 is set to the modulo-two addition of states on the feedback path (if $a_i = 1$ then x_i is added, if $a_i = 0$ then it is not).
- Output $u(k)$ is collected at state x_m.

Remark: such a feedback register is easily implemented in hardware.

Modulo-two addition

Formula/truth table of modulo-two addition:

$$p \oplus q = \begin{cases} 0 & \text{if } p = 0, q = 0 \\ 1 & \text{if } p = 0, q = 1 \\ 1 & \text{if } p = 1, q = 0 \\ 0 & \text{if } p = 1, q = 1 \end{cases}$$

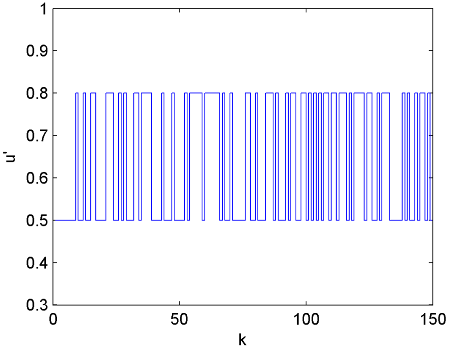
...also known as XOR (eXclusive OR)

Arbitrary-valued PRBS

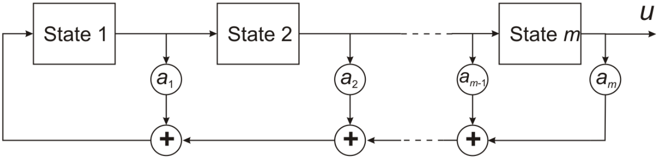
To obtain a signal $u'(k)$ taking values b, c instead of $0, 1$, shift & scale the original signal $u(k)$:

$$u'(k) = b + (c - b)u(k)$$

Example for $b = 0.5$, $c = 0.8$:



State space representation



$$
\begin{aligned}
x_1(k+1) &= a_1 x_1(k) \oplus a_2 x_2(k) \oplus \cdots \oplus a_m x_m(k) \\
x_2(k+1) &= x_1(k) \\
&\vdots \\
x_m(k+1) &= x_{m-1}(k) \\
u(k) &= x_m(k)
\end{aligned}
$$

$x(k) = [x_1(k), \ldots, x_m(k)]^\top$ compactly denotes the state vector of m variables (bits)

State space representation: matrix form

$$x(k+1) = \begin{bmatrix} a_1 & a_2 & \dots & a_{m-1} & a_m \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix} \otimes x(k) =: A \otimes x(k)$$

$$u(k) = [0\ 0\ \dots\ 0\ 1]x(k) =: Cx(k)$$

where $k \geq 0$, and $\otimes$ symbolically indicates that the additions in the matrix product are performed modulo 2.

Period of PRBS

- The PRBS algorithm is deterministic, so the current state $x(k)$ fully determines the future states and outputs

$\Rightarrow$ Period (number of steps until sequence repeats) at most 2^m

- The identically zero state is undesirable, as the future sequence would always remain 0

$\Rightarrow$ Maximum practical period is $P = 2^m - 1$

A PRBS with period $P = 2^m - 1$ is called maximum-length PRBS.

Such PRBS have interesting characteristics, so they are preferred in practice.

Maximum-length PRBS

The period is determined by the feedback coefficients a_i.

The following coefficients must be 1 to achieve maximum length (all others 0):

m	Max period $2^m - 1$	Coefficients equal to 1
3	7	a_1, a_3
4	15	a_1, a_4
5	31	a_2, a_5
6	63	a_1, a_6
7	127	a_1, a_7
8	255	a_1, a_2, a_7, a_8
9	511	a_4, a_9
10	1023	a_3, a_{10}

Other working combinations of coefficients exist, and coefficients for larger m can be found in the literature.

Matlab function

```
u = idinput(N, type, [], [b, c]);
```

Arguments:

1. `N`: signal length (number of discrete steps).
2. `type`: signal type, a string. Relevant for us: `'prbs'` for PRBS, `'rgs'` for white Gaussian noise, `'sin'` for multisine.
3. Third argument: the frequency band of the inputs (can be left at its default, empty matrix).
4. `[b, c]`: the range (lower and upper limits) of the signal. For Gaussian noise, $[b, c]$ is instead the one-standard-deviation interval below and above the mean.

Remark: `N` can be configured to generate multiple-input signals (see the Matlab documentation for details).

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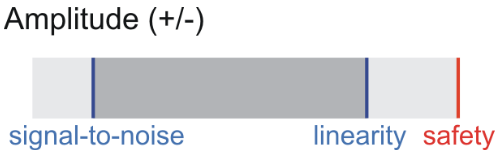
Choice of input shape

Some identification methods require specific types of inputs:

- Transient analysis requires step or impulse inputs.
- Correlation analysis preferably works with white-noise input.

Rule of thumb: input shapes, including characteristics like amplitude, should be chosen to be representative for the typical operation of the system

Choice of input amplitude



- Range of allowed inputs typically constrained by system operator, due to safety or cost concerns
- Even if allowed, overly large inputs may take the system out of its zone of linearity and lead to poor performance of linear identification
- But too small inputs will lead to signals dominated by noise and disturbance

Choice of sampling interval

For nearly all methods, we work in discrete time so we must choose a sampling interval T_s

- Too large intervals will not model the relevant dynamics of the system. Initial idea: 10% of the smallest time constant
- Too small intervals will lead to overly large effects of noise and disturbance
- When in doubt, take T_s smaller

Due to Nyquist-Shannon, we know that signals cannot be recovered above frequency $1/(2T_s)$, so to mitigate noise and other effects it is useful to pass the outputs (and inputs, if measured) through a low-pass filter that eliminates higher frequencies

Mean and covariance

Given a random signal $u(k)$, its mean and covariance are defined:

$$\mu = \mathrm{E}\left\{u(k)\right\}$$
$$r_u(\tau) = \mathrm{E}\left\{[u(k+\tau) - \mu][u(k) - \mu]\right\}$$

Notes:

- Recall mean and variance of random variables
- The same covariance function $r_u(\tau)$ was used in correlation analysis, where we assumed the signal is zero-mean
- Zero-mean signals may work better even for other methods, like ARX

Mean and covariance: deterministic signal

When the signal is deterministic (e.g. PRBS), the mean and covariance are redefined as:

$$\mu = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} u(k)$$

$$r_u(\tau) = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} [u(k+\tau) - \mu][u(k) - \mu]$$

Note: $\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \cdot$ is the same as $\mathrm{E}\{\cdot\}$ for a (well-behaved) random signal.

Persistent excitation

Even methods that do not fix the input shape make requirements on the inputs: e.g. for ARX we required that $u(k)$ is "sufficiently informative", without making that property formal

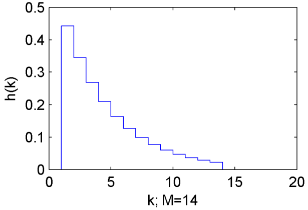
This condition can be precisely stated in terms of a property called persistence of excitation

Persistent excitation: Motivating example

We develop an *idealized* version of correlation analysis. This is only an intermediate motivating step, and the property is useful in many identification algorithms.

Finite impulse response (FIR) model:

$$y(k) = \sum_{j=0}^{M-1} h(j)u(k-j) + v(k)$$



Correlation analysis: Covariances

Assuming $u(k)$, $y(k)$ are zero-mean, so the means do not need to be subtracted:

$$r_u(\tau) = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} u(k+\tau)u(k)$$

$$r_{yu}(\tau) = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} y(k+\tau)u(k)$$

In practice covariances must be estimated from finite datasets, but here we work with their ideal values (since this is only a motivating example, which we do not actually implement).

Correlation analysis: Identifying the FIR

Taking M equations to find the FIR parameters, we have:

$$\begin{bmatrix} r_{yu}(0) \\ r_{yu}(1) \\ \vdots \\ r_{yu}(M-1) \end{bmatrix} = \boxed{\begin{bmatrix} r_u(0) & r_u(1) & \dots & r_u(M-1) \\ r_u(1) & r_u(0) & \dots & r_u(M-2) \\ \vdots & & & \\ r_u(M-1) & r_u(M-2) & \dots & r_u(0) \end{bmatrix}} \cdot \begin{bmatrix} h(0) \\ h(1) \\ \vdots \\ h(M-1) \end{bmatrix}$$

We are allowed to take a square system (number of equations equal to number of parameters) because we are in the idealized, noise-free case, so overfitting is not a concern.

Denote the matrix in the equation by $R_u(M)$, the covariance matrix of the input.

Persistent excitation: formal definition

Definition

A signal $u(k)$ is persistently exciting (PE) of order n if $R_u(n)$ is positive definite.

A matrix $A \in \mathbb{R}^{n \times n}$ is positive definite if $h^\top A h > 0$ for any nonzero vector $h \in \mathbb{R}^n$. Note that A must be nonsingular.

Examples:

- $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is positive definite. Denote $h = \begin{bmatrix} a \\ b \end{bmatrix}$, then $h^\top A h = a^2 + b^2$.
- $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ is not positive definite. Counterexample: $h = \begin{bmatrix} a \\ -a \end{bmatrix}$, $h^\top A h = -2a^2$.

PE in correlation analysis

If the order of PE is M, then $R_u(M)$ is positive definite, hence invertible and the linear system from correlation analysis can be solved to find an FIR of length M.

So an order M of PE means that an FIR model of length M is identifiable (M parameters can be found).

General role of PE

Beyond FIR, PE plays a role in *all* parametric system identification methods, including ARX and methods still to be discussed, like prediction error methods and instrumental variable techniques.

A large enough order of PE is required to properly identify the parameters.

Typically, the required order is a multiple of (e.g. twice) the number of parameters n that must be estimated.

Covariance alternatives

In the sequel we will always use the following, simpler definition:

$$r_u(\tau) = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} u(k+\tau)u(k)$$

even when u is nonzero-mean. Even though in that case r_u is no longer the true covariance in the statistical sense, it is still useful.

When applying the PE condition for nonzero-mean signals, the simplified definition above will lead to an order of PE larger by 1 than the order of PE obtained with the means removed.

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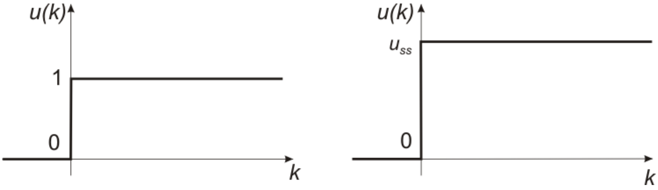
Step input



Take the more general, non-unit step:

$$u(k) = \begin{cases} 0 & k < 0 \\ u_{ss} & k \geq 0 \end{cases}$$

Step input: Mean and covariance

Mean and covariance:

$$\mu = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} u(k) = u_{ss}$$

$$r_u(\tau) = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} u(k+\tau)u(k) = u_{ss}^2$$

Note the signal starts from $k = 0$, so the summation is modified (unimportant to the final result).

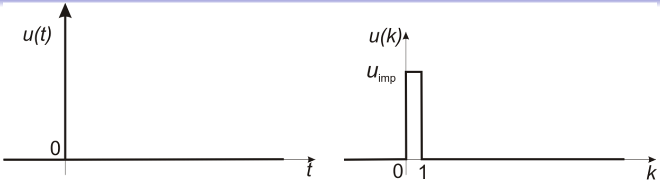
Step input: Order of PE

Covariance matrix:

$$R_u(n) = \begin{bmatrix} r_u(0) & r_u(1) & \dots & r_u(n-1) \\ r_u(1) & r_u(0) & \dots & r_u(n-2) \\ \vdots & & & \\ r_u(n-1) & r_u(n-2) & \dots & r_u(0) \end{bmatrix} = \begin{bmatrix} u_{ss}^2 & u_{ss}^2 & \dots & u_{ss}^2 \\ u_{ss}^2 & u_{ss}^2 & \dots & u_{ss}^2 \\ \vdots & & & \\ u_{ss}^2 & u_{ss}^2 & \dots & u_{ss}^2 \end{bmatrix}$$

This matrix has rank 1, so a step input is PE of order 1.

Impulse input



Recall discrete-time realization:

$$u(k) = \begin{cases} \frac{1}{T_s} & k = 0 \\ 0 & \text{otherwise} \end{cases}$$

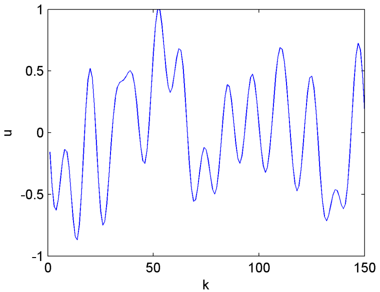
Mean and covariance:

$$\mu = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} u(k) = 0$$

$$r_u(\tau) = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} u(k+\tau)u(k) = 0$$

$\Rightarrow R_u(n)$ matrix of zeros, the impulse is not PE of any order.

Sum of sines



$$u(k) = \sum_{j=1}^{m} a_j sin(\omega_j k + \varphi_j), \quad 0 \leq \omega_1 < \omega_2 < \ldots < \omega_n \leq \pi$$

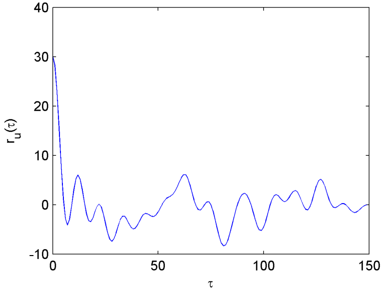
Mean and covariance:

$$\mu = \begin{cases} a_1 sin(\varphi_1) & \text{if } \omega_1 = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$r_u(\tau) = \sum_{j=1}^{m-1} \frac{a_j^2}{2} \cos(\omega_j \tau) + \begin{cases} a_m^2 \sin^2 \varphi_m & \text{if } \omega_m = \pi \\ \frac{a_m^2}{2} \cos(\omega_m \tau) & \text{otherwise} \end{cases}$$

Sum of sines (continued)

For the multisine exemplified before, the covariance function is:



A multisine having m components is PE of order n with:

$$n = \begin{cases} 2m & \text{if } \omega_1 \neq 0, \omega_m \neq \pi \\ 2m-1 & \text{if } \omega_1 = 0 \text{ or } \omega_m = \pi \\ 2m-2 & \text{if } \omega_1 = 0 \text{ and } \omega_m = \pi \end{cases}$$

White noise: Mean and covariance

Take a zero-mean white noise signal of variance σ^2, e.g. drawn from a Gaussian:

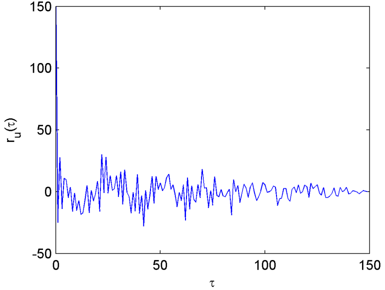
$$u(k) \sim \mathcal{N}(0, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right)$$

Then, by definition:

$$\mu = 0$$
$$r_u(\tau) = \begin{cases} \sigma^2 & \text{if } \tau = 0 \\ 0 & \text{otherwise} \end{cases}$$

White noise: Covariance example

Covariance function of white noise signal exemplified before:



White noise: Order of PE

Covariance matrix:

$$R_u(n) = \begin{bmatrix} r_u(0) & r_u(1) & \dots & r_u(n-1) \\ r_u(1) & r_u(0) & \dots & r_u(n-2) \\ \vdots & & & \\ r_u(n-1) & r_u(n-2) & \dots & r_u(0) \end{bmatrix}$$

$$= \begin{bmatrix} \sigma^2 & 0 & \dots & 0 \\ 0 & \sigma^2 & \dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots & \sigma^2 \end{bmatrix} = \sigma^2 I_n$$

where I_n = the identity matrix, positive definite.

$\Rightarrow$ for any n, $R_u(n)$ positive definite — white noise is PE of any order.

Question

Given the information above, why does correlation analysis prefer white noise to other input signals, in order to identify the FIR?

PRBS: Mean

Consider a 0, 1-valued, maximum-length PRBS with m bits: $P = 2^m - 1$, a large number.

Then its state $x(k)$ will contain all possible binary values with m digits except 0.

Signal $u(k)$ is the last position of $x(k)$, which takes value 1 a number of 2^{m-1} times, and value 0 a number of $2^{m-1} - 1$ times.

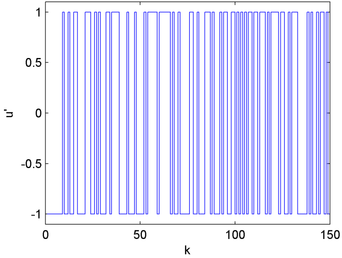
$\Rightarrow$ Mean value:

$$\mu = \frac{0}{P-1}\sum_{k=1}^{P} u(k) = \frac{1}{P}2^{m-1} = \frac{(P+1)/2}{P} = \frac{1}{2} + \frac{1}{2P} \approx \frac{1}{2}$$

where the approximation holds for large P.

PRBS: Covariance

Consider a zero-mean PRBS, scaled between $-b$ and b:



$$u'(k) = -b + 2bu(k)$$

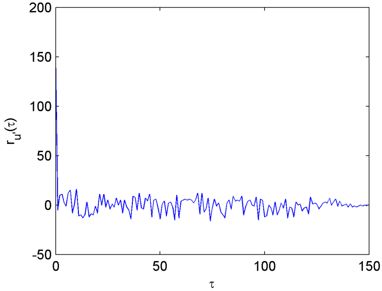
Then:

$$\mu = -b + 2b(\frac{1}{2} + \frac{1}{2P}) = \frac{b}{P} \approx 0$$

$$r_u(\tau) = \begin{cases} 1 - \frac{1}{P^2} \approx 1 & \text{if } \tau = 0 \\ -\frac{1}{P} - \frac{1}{P^2} \approx -\frac{1}{P} \approx 0 & \text{otherwise} \end{cases}$$

PRBS: Covariance example

Covariance function of the zero-mean PRBS above:



So, PRBS behaves similarly to white noise (similar covariance function). Combined with the ease of generating it, this property makes PRBS very useful in system identification.

PRBS: Order of PE

A maximum-length PRBS is PE of exactly order P, the period (and not larger).

Exercise

Take a small value of $P \geq 2$ and, using the formula for the covariance function of the PBRS, show that the PRBS is exactly of PE order P.
Hint: construct $R_u(n)$ for $n = P$ and show that it is rank P, then for $n > P$ and show it is *still* only of rank P. This can be done by showing that columns $P+1, P+2, \ldots$ are linear combinations of the first P columns.

Summary

- Common input signals: step, impulse, multisine, zero-mean white noise, pseudo-random binary sequence
- PRBS details: generation using LSFRs, maximal period
- Choosing input amplitude and sampling period
- Mean and covariance of input signals
- Order of persistent excitation
- Characterizing mean, covariance, and PE order for all common input signals