System Identification

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Part X

Model validation and practical issues

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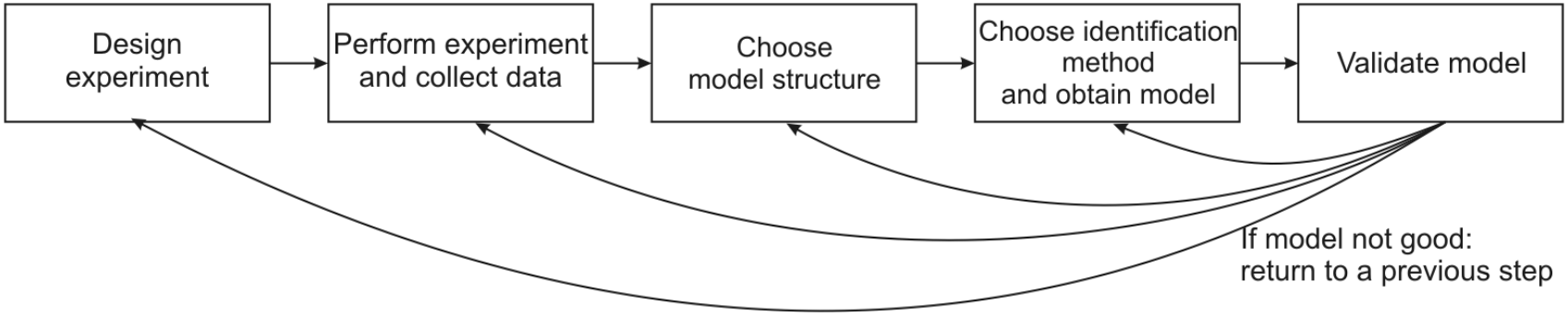
Recall: Importance of validation



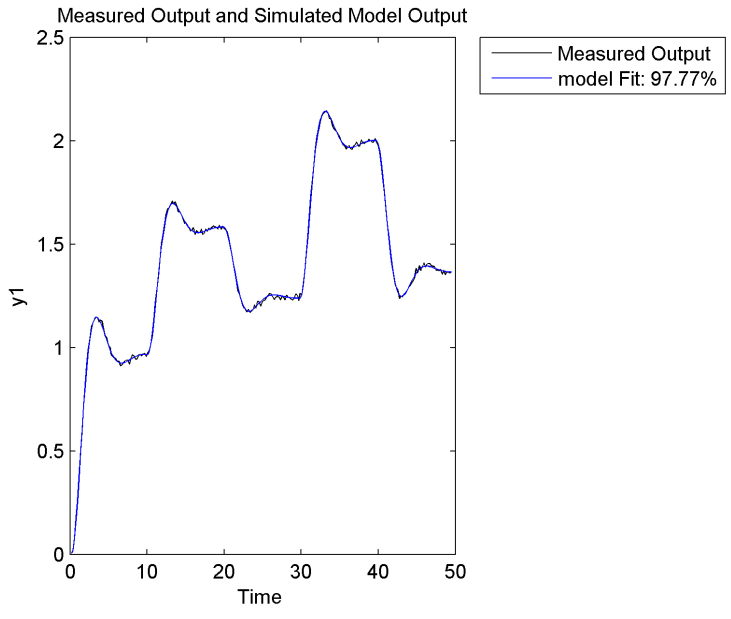
Model validation is a crucial step: the model must be good enough (for its intended usage).

If validation is unsuccessful, previous steps in the workflow must be redone, for instance:

- Rerun the identification algorithm with different parameters (e.g. δ in recursive methods).
- Change the model structure: e.g. orders of polynomials *na*, *nb* in ARX, or even the model type entirely, e.g. IV instead of ARX
- Design and run a new experiment (e.g. more data, different input signal)

Motivation

So far, we validated and selected models mostly informally, by examining plots or comparing errors – using *common sense*.



Next, some mathematically well-founded tests will be given.

However, common sense remains indispensable – mathematical tests work under assumptions that may not always be satisfied.

Focus: Prediction error methods

We focus on *single-output* models obtained by *prediction error methods*.

Some of the tests can be extended to other settings.

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Whiteness: Intuition

Recall general PEM model structure:

$$y(k) = G(q^{-1})u(k) + H(q^{-1})e(k)$$

where $e(k)$ is *assumed* to be zero-mean white noise.

PEM are derived so that the prediction error $\varepsilon(k) = y(k) - \widehat{y}(k) = e(k)$. If the system satisfies the model structure (so the white-noise assumption holds), and moreover if the model is accurate, then $\varepsilon(k)$ is also zero-mean white noise.

Whiteness hypothesis

(W) The prediction errors $\varepsilon(k)$ are zero-mean white noise.

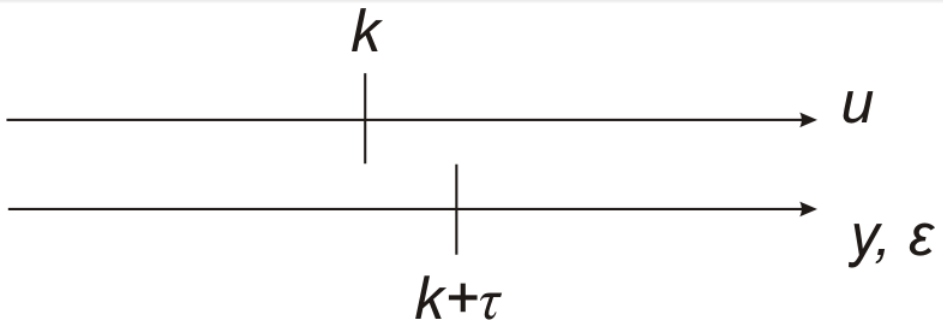
Independence of past inputs: Intuition

$$y(k) = G(q^{-1})u(k) + v(k)$$

If the model G is accurate, it entirely explains the influence of inputs $u(k)$ on current and future outputs $y(k+\tau)$. Therefore, the errors $\varepsilon(k+\tau) = y(k+\tau) - \widehat{y}(k+\tau)$ are only influenced by the disturbances v, and are *independent* of inputs $u(k)$. This holds regardless of whether the whiteness hypothesis is true or not.

Independence hypothesis 1

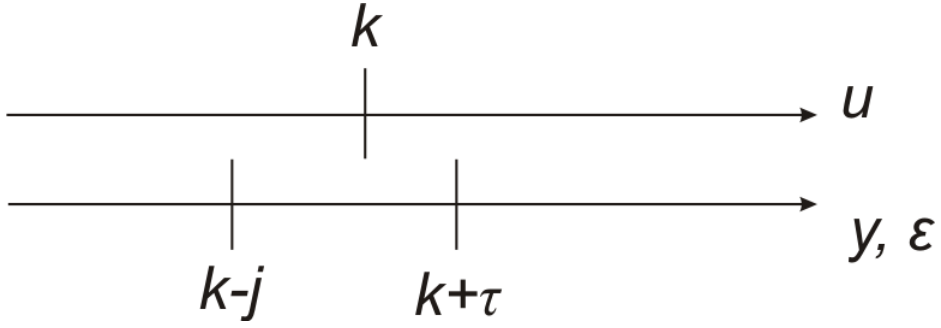
(I1) The prediction errors $\varepsilon(k+\tau)$ are independent of inputs $u(k)$ for $\tau \geq 0$ (i.e., current and future errors are independent of current inputs).



Independence of all inputs: Intuition

$$y(k) = G(q^{-1})u(k) + v(k)$$

If the experiment is closed-loop, $u(k)$ depends on past outputs and this will lead to a correlation of past errors $\varepsilon(k+\tau), \tau < 0$ with $u(k)$ (note the independence for $\tau \geq 0$ is not affected). If open-loop, then $\varepsilon(k+\tau), \tau < 0$ is also independent from $u(k)$.



Independence hypothesis 2

(I2) The prediction errors $\varepsilon(k+\tau)$ are independent of $u(k)$ for *any* τ (i.e., all the errors are independent of all the inputs).

All hypotheses

(W) The prediction errors $\varepsilon(k)$ are zero-mean white noise.

(I1) The prediction errors $\varepsilon(k+\tau)$ are independent of $u(k)$ for $\tau \geq 0$ (current and future errors are independent of current inputs).

(I2) The prediction errors $\varepsilon(k+\tau)$ are independent of $u(k)$ for *any* τ (all the errors are independent of all the inputs).

A good model should satisfy (W) and (I1), and if there is no feedback, also (I2).

We will develop tests that allow to either accept or reject these hypotheses for a given model, and therefore validate or reject the model.

Whiteness: Correlations

Recall the correlation function (equal to the covariance in zero-mean case):

$$r_\varepsilon(\tau) = \mathrm{E}\left\{\varepsilon(k+\tau)\varepsilon(k)\right\}$$

If $\varepsilon(k)$ is zero-mean white noise:

- The correlation function is zero, $r_\varepsilon(\tau) = 0$ for any nonzero τ.
- At zero, $r_\varepsilon(0)$ is the variance σ^2 of the white noise.

Whiteness: Correlations from data

Correlations are estimated from data, and then normalized by the (estimated) variance:

$$\widehat{r}_\varepsilon(\tau) = \frac{1}{N}\sum_{k=1}^{N-\tau} \varepsilon(k+\tau)\varepsilon(k)$$

$$x(\tau) = \frac{\widehat{r}_\varepsilon(\tau)}{\widehat{r}_\varepsilon(0)}$$

Normalization helps since we can think of the normalized magnitudes independently from any system details, whereas the unnormalized magnitudes depend of the nature of the system and signal (mV, cells per milliliter, m, km all lead to different numbers).

Whiteness test

In practice, $x(\tau)$ will never be zero for finite data, so instead we check if it is small for nonzero τ. For statistical reasons, we impose a cutoff at $\frac{1.96}{\sqrt{N}}$

Whiteness test

If $|x(\tau)| \leq \frac{1.96}{\sqrt{N}}$ for all $\tau \neq 0$ supported by the data, then the whiteness hypothesis (W) is accepted. Otherwise, (W) is rejected.

Independence: Correlations & their computation from data

To verify independence of ε from u, use *cross-correlation* function:

$$r_{\varepsilon u}(\tau) = \mathrm{E}\left\{\varepsilon(k+\tau)u(k)\right\}$$

1. If (I1) is true, then $r_{\varepsilon u}(\tau) = 0$ for $\tau \geq 0$.
2. If (I2) is true, then $r_{\varepsilon u}(\tau) = 0$ for any τ.

Estimation from data and normalization:

$$\widehat{r}_{\varepsilon u}(\tau) = \begin{cases} \frac{1}{N}\sum_{k=1}^{N-\tau} \varepsilon(k+\tau)u(k) & \text{if } \tau \geq 0 \\ \frac{1}{N}\sum_{k=1-\tau}^{N} \varepsilon(k+\tau)u(k) & \text{if } \tau < 0 \end{cases}$$

$$x(\tau) = \frac{\widehat{r}_{\varepsilon u}(\tau)}{\sqrt{\widehat{r}_{\varepsilon}(0)\widehat{r}_{u}(0)}}$$

Independence test at τ

Independence tests

If $|x(\tau)| \leq \frac{1.96}{\sqrt{N}}$, $\forall \tau \geq 0$ supported by the data, then the independence hypothesis (I1) is accepted.
If the condition holds $\forall \tau$ supported by the data (including negative τ), then (I2) is also accepted.

If the model is accurate (I1 holds), then checking the condition at $\tau < 0$ (I2) verifies the presence of feedback.

Correlation tests: Overall interpretation

$$y(k) = G(q^{-1})u(k) + H(q^{-1})e(k)$$

- If W and I1 hold, then the entire model (G and H) is correct
- If I1 holds but W fails, then G is correct but H is incorrect
- If I1 holds and I2 fails, there is feedback in the data. If I2 also holds then there is no feedback
- If I1 fails, then G is incorrect and there is not much else that we can conclude

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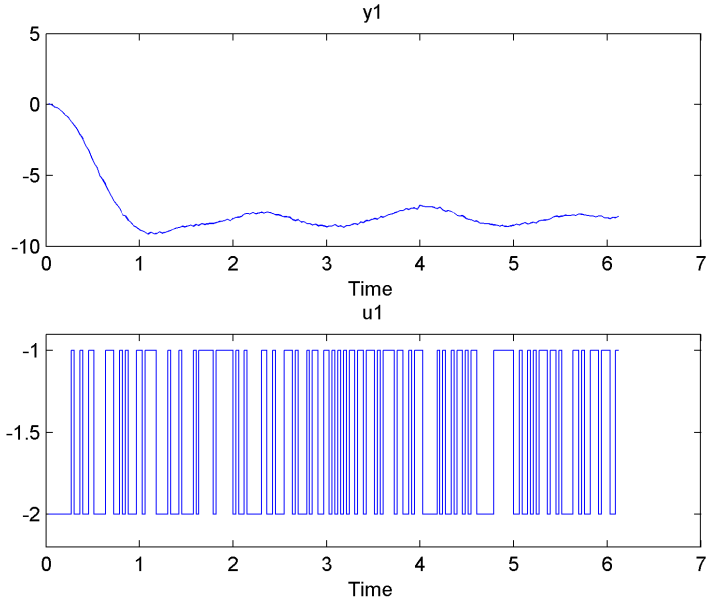
Matlab example: Experimental data

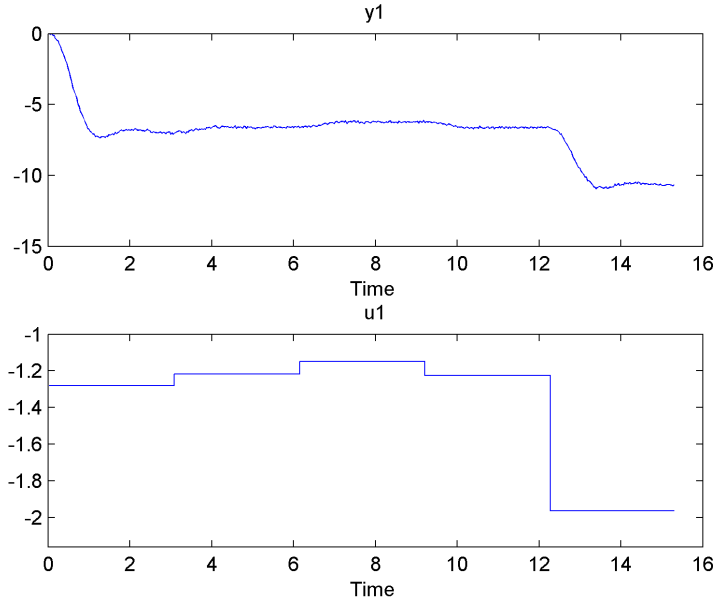
The real system is in output-error form:

$$y(k) = \frac{B(q^{-1})}{F(q^{-1})}u(k) + e(k)$$

and has order $n = 3$.

`plot(id);` and `plot(val);`

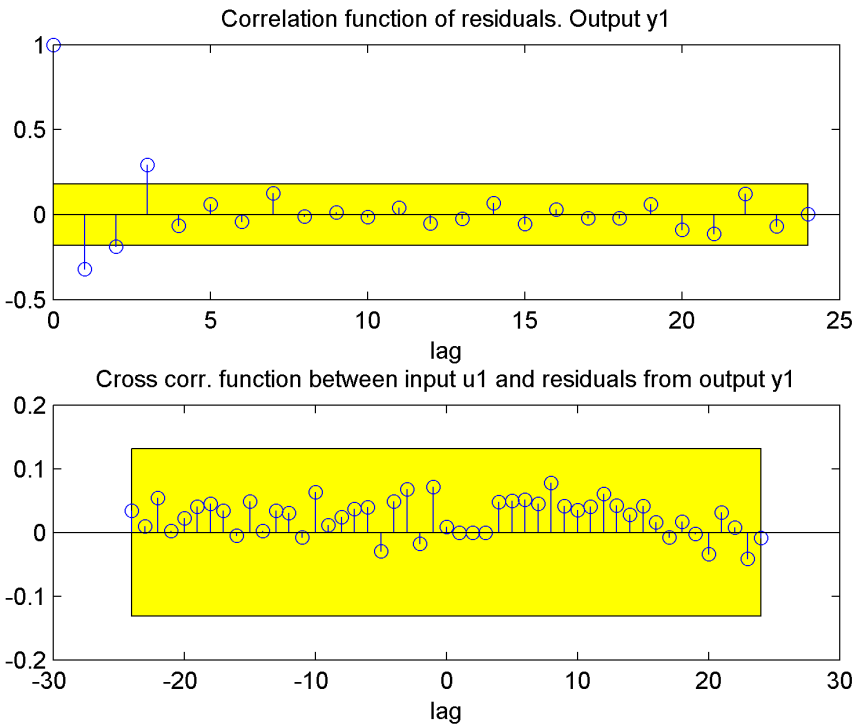




Matlab: ARX model

First, we try an ARX model:

```
mARX = arx(id, [3, 3, 1]); resid(mARX, id);
```

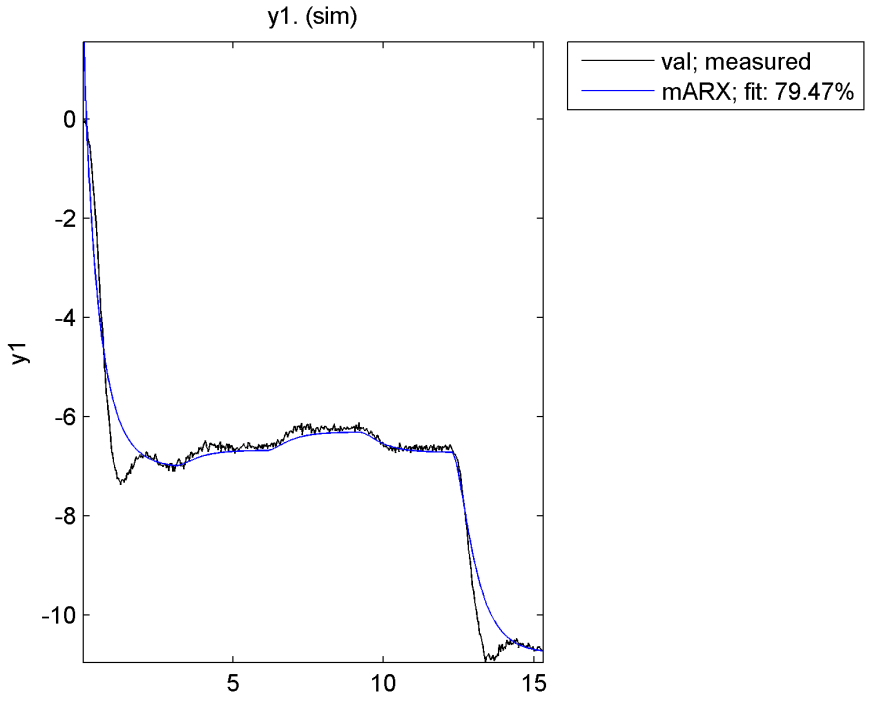


The whiteness test (W) fails, and the model is rejected. This is because the system is not within the model class.

As I1 holds, we conclude that the input-output model G is good, but the noise model H is wrong and we should work to improve that part.

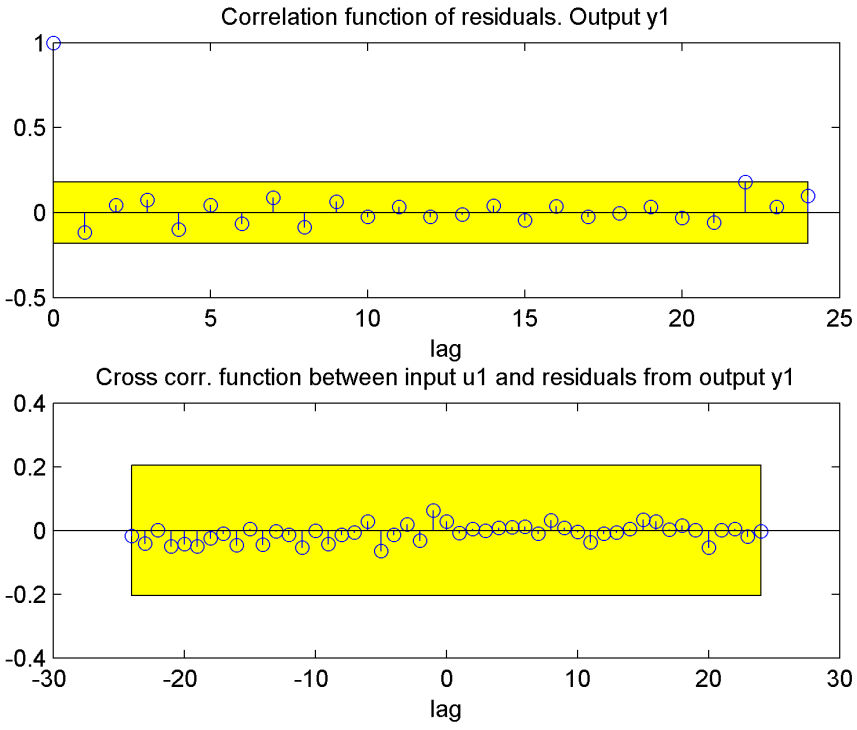
Matlab: ARX model (continued)

Simulating on the validation data confirms the fact that the model is poor.



Matlab: OE model

```
mOE = oe(id, [3, 3, 1]); resid(mOE, id);
```

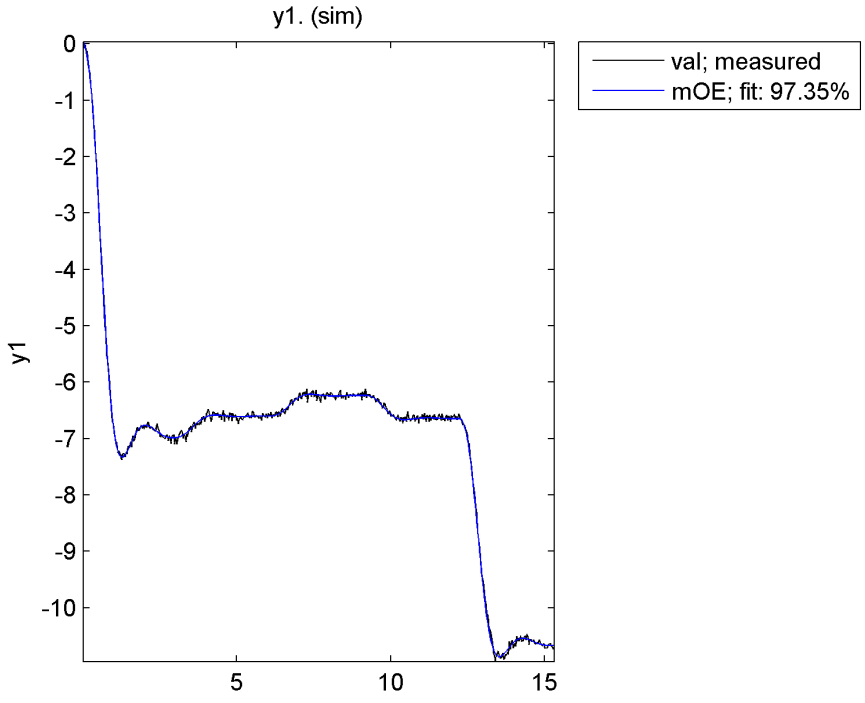


OE model passes all the tests – as expected because the OE model class contains the real system. Thus, both G and H are validated.

Important note: The Matlab functions impose a smaller cutoff for the correlations, so they are less likely to reject a correct model.

Matlab: OE model (continued)

Simulating the model on the validation data confirms the model has good quality.



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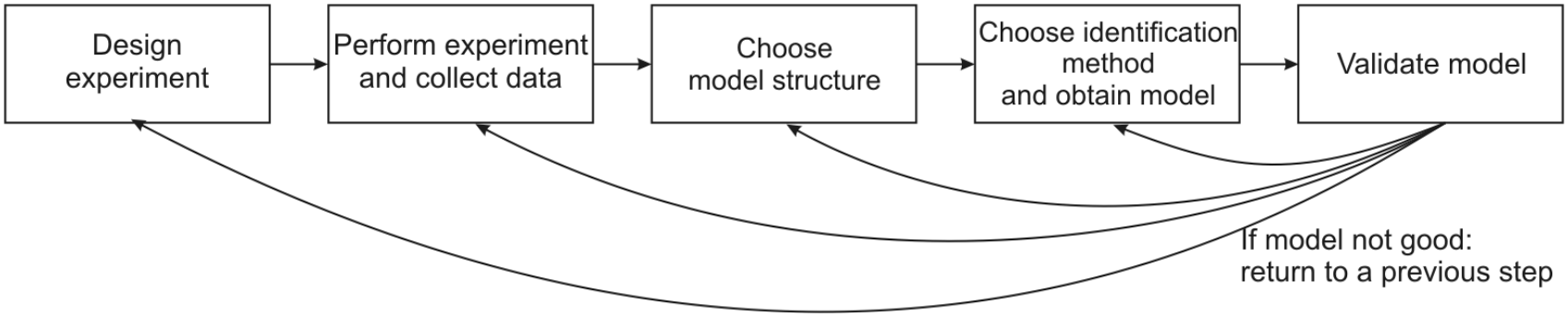
Structure selection in workflow



While we nearly always tuned the model structure (e.g. type, orders, length), the criteria for doing so were often informal.

Next, we discuss structure selection in a formal way.

Structure selection: Model complexity

Consider we are given several model structures $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_\ell$.
Example: ARX structures of increasing order.

How to choose among them?

First idea: choose $\mathcal{M}_i$ leading to the smallest mean squared error:

$$V(\widehat{\theta}) = \frac{1}{N} \sum_{k=1}^{N} \varepsilon(k)^2$$

This ignores the complexity of the model, which is related to:

- the computational effort for identification and simulation
- the amount of data needed for identification
- the risk of overfitting

We explore other options that do consider model complexity (without going into their derivation).

Akaike's information criterion (AIC)

$$W_{\mathrm{AIC}} = N \log V(\widehat{\theta}) + 2p, \text{ or equivalently: } \log V(\widehat{\theta}) + \frac{2p}{N}$$

where N is the number of data points and p the number of parameters (e.g., $na + nb$ in ARX).

Choice: Model with smallest W_{AIC}.

Intuition:

- The term $2p$ penalizes the complexity of the model (number of parameters).
- Division by the number N of data points in $2p/N$ takes into account that more data allows more parameters to be identified.
- Taking the logarithm of the MSE allows to better differentiate between small values of the MSE.

Final prediction error (FPE)

$$W_{\text{FPE}} = V(\widehat{\theta})\frac{1 + p/N}{1 - p/N}$$

Choice: Model with smallest W_{FPE}.

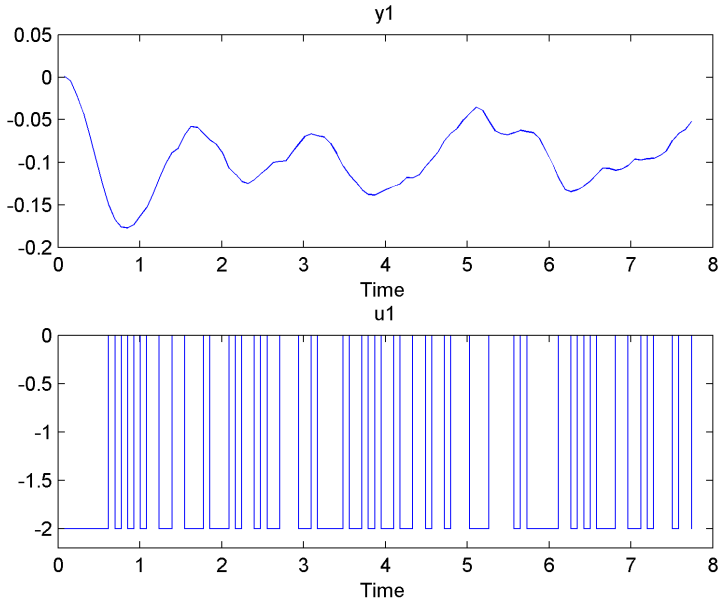
Intuition: When N is large:

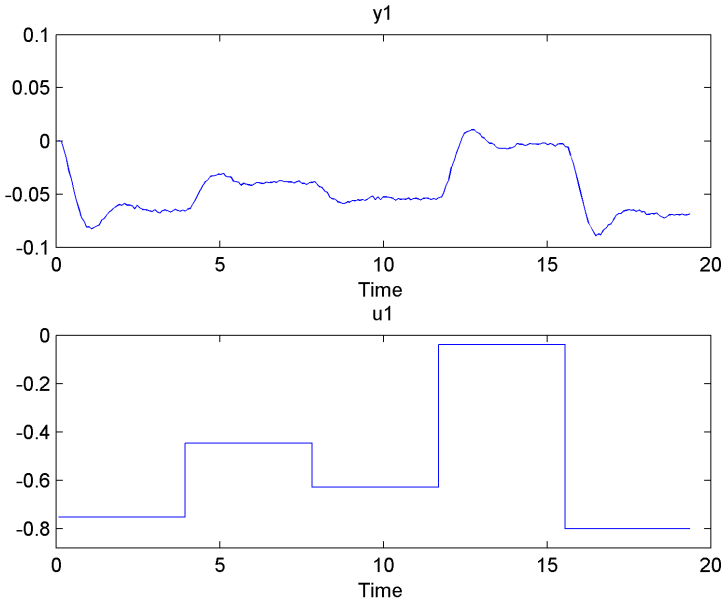
$$V(\widehat{\theta})\frac{1 + p/N}{1 - p/N} = V(\widehat{\theta})(1 + \frac{2p/N}{1 - p/N}) \approx V(\widehat{\theta})(1 + \frac{2p}{N})$$

and the term $\frac{2p}{N}$ works like before, but now it leads to a correction proportional to the MSE rather than getting added directly.

Matlab example

An OE system with $n = 2$.





Matlab: `selstruc` with AIC

Recall `arxstruc`:

```
Na = 1:15; Nb = 1:15; Nk = 1:5;
NN = struc(Na, Nb, Nk); V = arxstruc(id, val, NN);
```

- `struc` generates all combinations of orders in `Na, Nb, Nk`.
- `arxstruc` identifies for each combination an ARX model on the data `id`, simulates it on the data `val`, and returns information about the MSEs, model orders etc. in `V`.

Matlab: `selstruc` with AIC (continued)

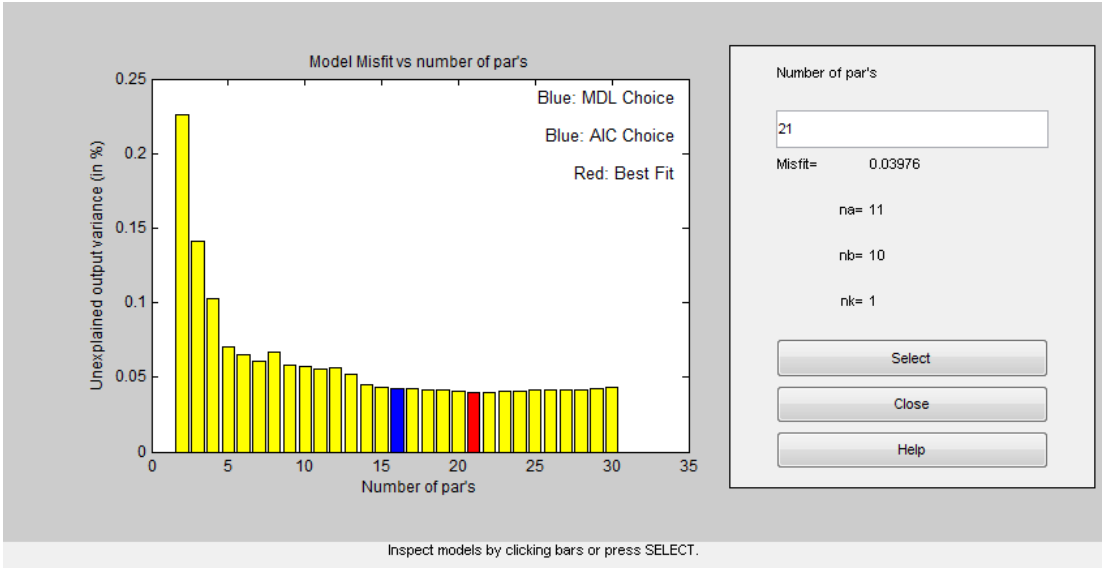
To choose the structure with the Akaike's information criterion:

```
N = selstruc(V, 'aic');
```

For our data, `N`= [8, 8, 1].

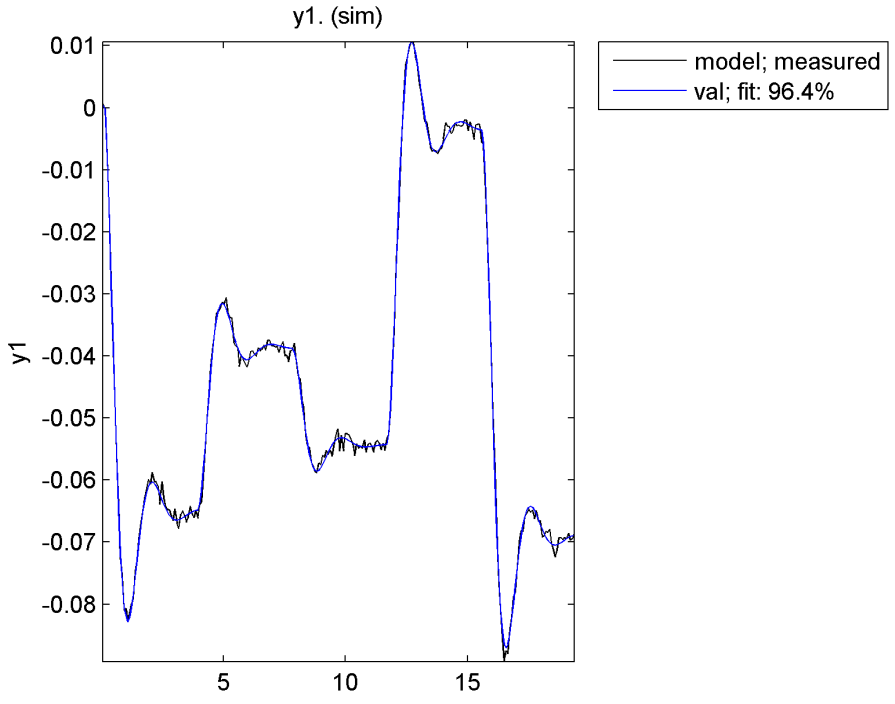
Alternatively, graphical selection also allows using AIC:

```
N = selstruc(V, 'plot');
```



Note that the best-AIC model is not (always) the same as the best-fit model!

Matlab: Results



Remarks

AIC, FPE also work if the system is not in the model class.

Matlab offers functions `aic`, `fpe` that compute these criteria for a list of models with any structure.

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Motivation

Consider a case where *the real system* obeys the ARMAX structure:

$$A_0(q^{-1})y(k) = B_0(q^{-1})u(k) + C_0(q^{-1})e(k)$$

where subscript 0 indicates quantities related to the real system.

This is equivalent to any model:

$$W(q^{-1})A_0(q^{-1})y(k) = W(q^{-1})B_0(q^{-1})u(k) + W(q^{-1})C_0(q^{-1})e(k)$$

with $W(q^{-1})$ some polynomial of order nw.

So, using ARMAX identification with $na = na_0 + nw$, $nb = nb_0 + nw$, $nc = nc_0 + nw$ can produce an accurate model. This model is however too complicated (overparametrized), and will have some nearly common factors $W(q^{-1})$ in all polynomials (only "nearly" because of the approximate nature of the identification).

Pole-zero cancellations

This type of situation can be identified by checking if some poles and zeros of the model (approximately) cancel each other out.

We exemplify using Matlab function `pzmap`, which shows the poles and zeros of G in the generic model:

$$y(k) = G(q^{-1})u(k) + v(k)$$

For the ARMAX example, $G(q^{-1}) = \frac{W(q^{-1})B_0(q^{-1})}{W(q^{-1})A_0(q^{-1})}$, so the roots of W are both poles and zeros and (approximately) cancel each other out.

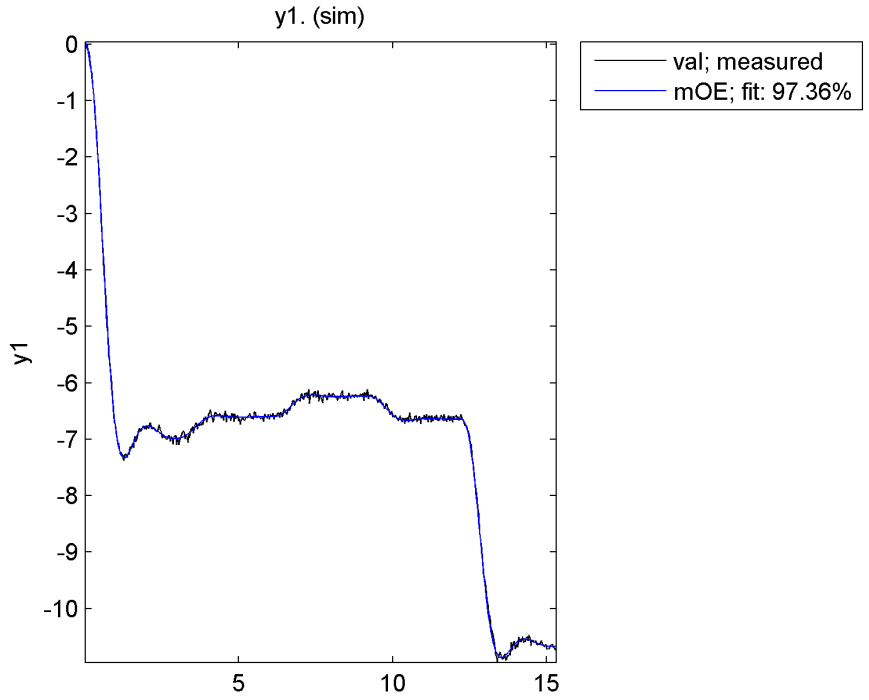
This idea extends to other model types besides ARMAX.

Matlab: overparameterized OE model

On the same data as for correlation tests (recall system has order $n = 3$):

```
mOE = oe(id, [5, 5, 1]);
```

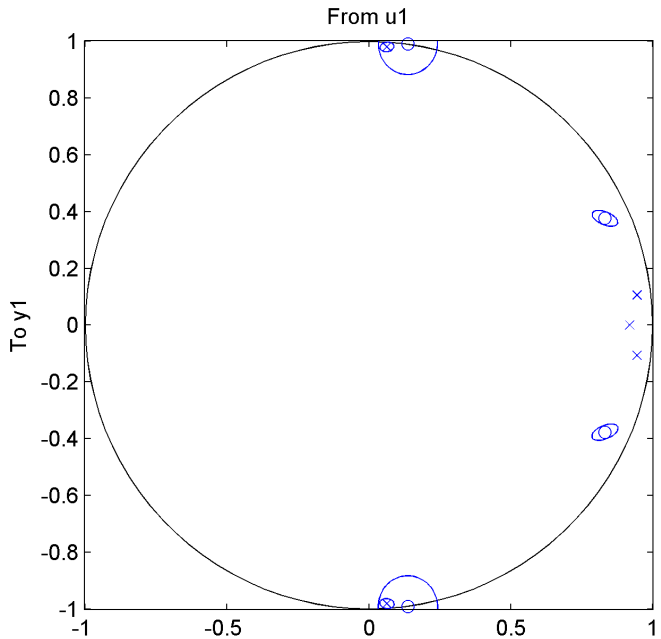
Looking at the validation data, the model is accurate:



Matlab: testing for pole-zero cancellations

```
pzmap(mOE, 'sd', nsd);
```

Arguments `'sd', nsd` specify a statistical confidence region around the poles and zeros. Here we take `nsd`=1.96, for statistical reasons.



Two pairs of poles and zeros have overlapping confidence regions $\Rightarrow$ likely they are canceling each other. This indicates that identification should be rerun with the true system order 3 (we already did this in our earlier OE results).

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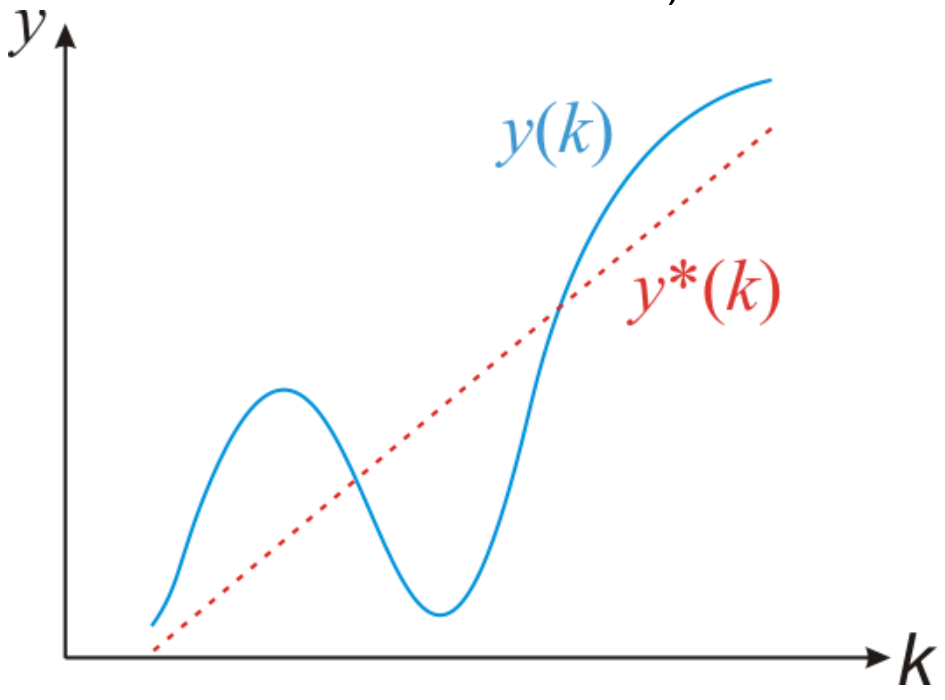
Drifts

Sometimes, the data will contain spurious slow signals called *drifts*, coming e.g. from slow disturbances (as opposed to the fast noise or disturbance, which we know how to handle)



Idea: Treat the drifts as time series, fit them with linear regression, and remove them

Estimating drifts

1. Treat the input and output as separate time series (no longer a dynamical system identification problem), write drift models:

$$u^*(k) = \theta_1^u + \theta_2^u k + \theta_3^u k^2 + \ldots + \theta_n^u k^{n-1}$$
$$y^*(k) = \theta_1^y + \theta_2^y k + \theta_3^y k^2 + \ldots + \theta_n^y k^{n-1}$$

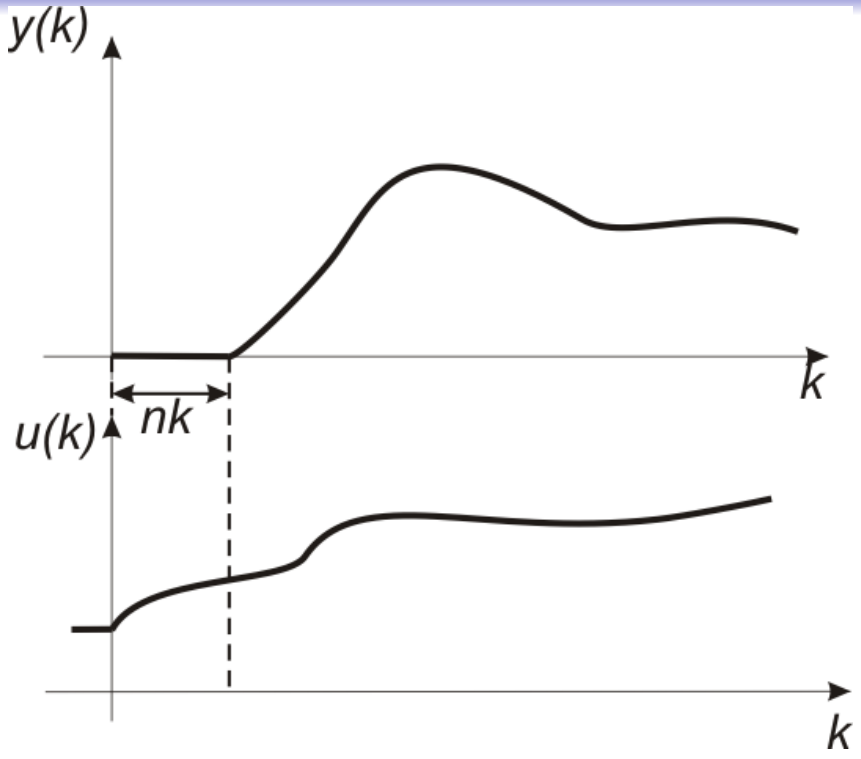
2. Find the parameter vectors θ^u, θ^y by linear regression on $u(k), y(k)$ and compute the corresponding drifts $u^*(k), y^*(k)$
3. Subtract the drifts from the data:

$$\bar{u}(k) = u(k) - u^*(k), \quad \bar{y}(k) = y(k) - y^*(k)$$

4. Identify as usual, but with "detrended" signals $\bar{u}, \bar{y}$

Notes: Matlab function `detrend` available; Removing zero-order drifts = removing the means

Time delays

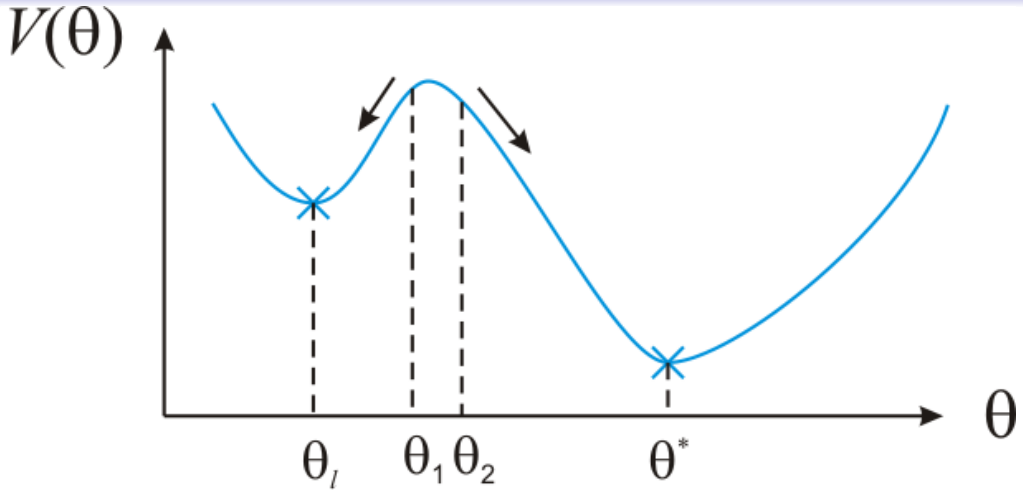


Read the delay on the graph. Set *nk* appropriately in Matlab, or otherwise add *nk* leading zeros to the polynomial $B(q^{-1})$ in the model:

$$\cdots y(k) = \frac{B(q^{-1})}{\cdots} u(k) + \cdots$$

Note: Taking *nk* too small is safe (possibly requiring an increase of *nb*); too large and it breaks the model!

Local minima

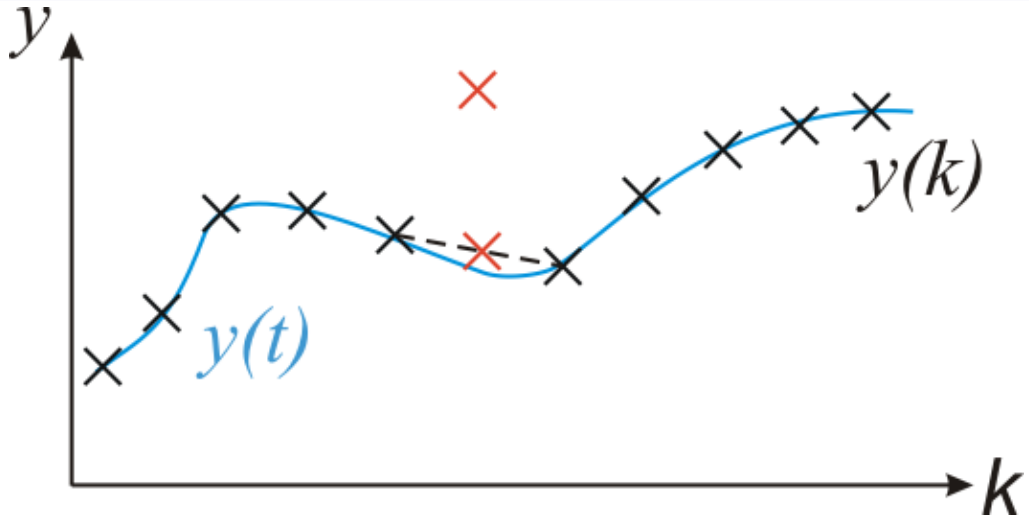


- Iterative optimization (needed for methods that cannot be solved as linear regression, such as ARMAX and OE) may get stuck in local minima
- E.g. if initialized to θ_1, Newton's method likely converges to the local minimum θ_ℓ. But from θ_2 it finds the global optimum θ^*!

$\Rightarrow$ If result is bad and local minima suspected, restart the optimization from another initial parameter vector

Note: ARMAX usually converges to the global optimum; OE often to local optima except when *u* is white noise

Outliers



- Sometimes, a few measurements will be wildly incorrect, due to e.g. transient malfunctions. These are called outliers
- Best tested via the prediction error ε, after finding an initial model: if $\varepsilon(k)$ is anomalously large at some step k, an outlier is likely
- Solution 1: Fill in the data using e.g. the average of $y(k-1)$ and $y(k+1)$ (shown in the figure), or the model prediction $\hat{y}(k)$
- Solution 2: Cap the prediction error at a reasonable maximum $\varepsilon_{\max}$, so $V(\theta) = \sum_{k=1}^{N} \min\{\varepsilon^2(k), \varepsilon_{\max}\}$