Optimistic Optimization

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1. Problem & motivation
2. DOO: Deterministic optimistic optimization
3. SOO: Simultaneous optimistic optimization
4. Application: Multiagent consensus
These methods were published in:


1 is original reference, 2 is an extensive survey including applications to control.
Optimization problem

\[
\max_{x \in X} f(x)
\]

Assumption 1

Function \( f : X \to \mathbb{R} \) is Lipschitz-continuous with respect to a semimetric \( \ell \):

\[
|f(x) - f(x')| \leq \ell(x, x')
\]

Definition (Semimetric)

A function \( \ell : X \times X \to \mathbb{R} \) satisfying:

- \( \ell(x, x') \geq 0 \)
- \( \ell(x, x') = \ell(x', x) \)
- \( \ell(x, x') = 0 \) if and only if \( x = x' \)

Intuitively: a notion of distance
Motivation

No method with guaranteed performance for any function

Especially when the metric $\ell$ is unknown
DOO idea

- Explore the space $X$ iteratively
- Always expand **optimistic** set, with largest upper bound: $b(X_i) = f(x_i) + \delta(X_i)$, diam. $\delta(X_i) = \sup_{x, x' \in X_i} \ell(x, x')$
- Until $n$ expansions exhausted
In general, a hierarchical partitioning rule must be defined.
- Set $X_{0,1} = X$ at depth 0 split into $X_{1,1}, \ldots, X_{1,K}$ at depth 1.
- Each set $X_{d,i}$ at depth $d$ split into $K$ subsets at depth $d + 1$.
### Assumption 2

The sets $X_{d,i}$ in the hierarchical partitioning must:

a) Shrink with the depth:
   \[ \delta(X_{d,i}) \leq \delta_d \text{ for any set } i \text{ at } d; \delta_d \text{ decreases with } d \]

b) Be well-shaped:
   each $\delta(X_{d,i})$ contains a ball in the semimetric $\ell$ having radius proportional to $\delta_d$, $B(x_{d,i}, \nu \delta_d)$
DOO algorithm

initialize tree with root $X_{0,1} = X$

for $t = 1$ to $n$ do

$X_{d,i}^\dagger \leftarrow \text{arg max}_{X_{d,i} \in \text{leaves}} b(X_{d,i})$

expand $X_{d,i}^\dagger$ (partition the set), adding children to tree

end for

output best sample $\hat{x}^* = \text{arg max}_{X_{d,i} \in \text{tree}} f(x_{d,i})$

(Munos, 2011)
Examples

- Quadratic function
- Rosenbrock banana function
An easy near-optimality guarantee

- Denote the expanded set at iteration $t$ by $X_t^+$
- $b(X_t^+) \geq f^*$, otherwise it wouldn’t have been selected
- $f(x_t^+) \leq f^*$ by definition
- $f(\hat{x}^*) \geq f(x_t^+)$ because $\hat{x}^*$ maximizes $f$ on the tree
- So $f^* - f(\hat{x}^*) \leq \delta(X_t^+)$ at any $t$, and therefore $\leq \delta_{d^*}$, where $d^*$ the deepest expanded depth
Near-optimality dimension

Definition (Near-optimality dimension)

Smallest $\beta$ so that the near-optimal sets:

$$X_\varepsilon = \{ x \in X | f^* - f(x) \leq \varepsilon \}$$

can be covered by (on the order of) $\varepsilon^{-\beta}$ balls of radius $\varepsilon$ in the semimetric $\ell$

$\beta$ measures how closely $\ell$ captures the smoothness of $f$
DOO near-optimality

**Theorem**

Consider a partition with exponentially decreasing sets, \( \delta_d = \gamma^d \), \( \gamma < 1 \). Then the solution returned by DOO satisfies:

\[
 f^* - f(\hat{x}^*) \approx \begin{cases} 
 n^{-1/\beta} & \text{if } \beta > 0 \\
 \gamma^{cn} & \text{if } \beta = 0 
\end{cases}
\]
Example: zero dimension

Take $f(x^*) - f(x) \approx |x^* - x|$ and $\ell(x, x') = |x - x'|$

$X_\varepsilon = \text{an interval of length } \varepsilon$, which is also an $\ell$-ball of size $\varepsilon$

So it takes a constant $= \varepsilon^0$ number of balls to cover $X_\varepsilon$, and $\beta = 0$

(taken from Munos)
Example: positive dimension

- If $f(x^*) - f(x) \approx |x^* - x|^2$, $X_\varepsilon$ is an interval of length $\sqrt{\varepsilon}$
- When $\ell(x, x') = |x - x'|^2$, a $\ell$-ball of size $\varepsilon$ is also an interval of length $\sqrt{\varepsilon}$, and $\beta = 0$
- When $\ell(x, x') = |x - x'|$, a $\ell$-ball of size $\varepsilon$ is an interval of length $\varepsilon$, so it takes $\varepsilon/\sqrt{\varepsilon} = \varepsilon^{-1/2}$ balls to cover $X_\varepsilon$, and $\beta = 1/2$
Example: semimetric mismatch

Influence of semimetric (mis)match for a quadratic function
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4. Application: Multiagent consensus
What if $\ell / \delta$ unknown? (i.e., smoothness of $f$ unknown)

Expand all potentially optimistic sets $X_{d,i}$, for which:

$$f(x_{d,i}) \geq f(x_{d',j})$$

for all leaves $j$ at smaller depths $d' \leq d$
SOO algorithm

initialize tree with root $X_{0,1} = X$

repeat at each iteration $t = 1, 2, \ldots$
  for $d = 0, \ldots, \min\{\text{current tree depth}, d_{\max}(t)\}$ do
    $X_{d,i}^\dagger \leftarrow \arg\max_{X_{d,i} \in \text{leaves at } d} f(x_{d,i})$
    if $f(x_{d,i}^\dagger) \geq f(x_{d',j})$ ∀ leaves $j$ at $d' \leq d$ then
      expand $X_{d,i}^\dagger$
    end if
  end for
until $n$ expansions performed
output best sample $\hat{x}^* = \arg\max_{x_{d,i} \in \text{tree}} f(x_{d,i})$

(Munos, 2011)
Examples

- Quadratic function
- Rosenbrock banana function
SOO near-optimality

**Theorem**

Consider a partition with exponentially decreasing sets, \( \delta_d = \gamma^d, \gamma < 1 \). Take \( d_{\text{max}}(t) = \sqrt{t} \), then the solution returned by SOO satisfies:

\[
 f^* - f(\hat{x}^*) \approx \begin{cases} 
 n^{-\frac{1}{2\beta}} & \text{if } \beta > 0 \\
 \gamma c' n & \text{if } \beta = 0 
\end{cases}
\]
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Consensus in nonlinear multiagent systems

- Agents with nonlinear dynamics \( x_{i,k+1} = f_i(x_{i,k}, u_{i,k}) \)
- **Consensus problem**: agents must reach agreement on (some) state variables
- Communication on an incomplete graph

**Challenge**: No solution for general \( f \)
Problem & motivation

DOO

Application

S0O

OO for consensus

1. Design target states with a classical consensus method
2. Use DOO or SOO to optimize action sequences in order to reach within $\varepsilon$ of target states

- **Consensus guaranteed** under conditions on $f$
- Tradeoff: length of action sequence must be known and small
Consensus of multiple robot arms