System Identification

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Part IV

Correlation analysis

Motivation 1

Why other techniques than transient analysis?

Transient analysis of step and impulse responses:

- Only works for a few system orders
- Must usually be done (semi-)manually
- Gives a rough, heuristic model of the system

The upcoming system identification methods:

- Work for arbitrary system orders
- Provide fully implementable, automatic algorithms
- Have solution accuracy guarantees (under appropriate conditions)

Motivation 2

Why correlation analysis?

- Closest to transient analysis (model = impulse response)
- True nonparametric model
- “Simple” general identification technique

Classification

Recall **taxonomy of models** from Part I:

By number of parameters:

1. Parametric models: have a fixed form (mathematical formula), with a known, often small number of parameters
2. Nonparametric models: cannot be described by a fixed, small number of parameters
   Often represented as graphs or tables

By amount of prior knowledge ("color"):

1. First-principles, white-box models: fully known in advance
2. Black-box models: entirely unknown
3. Gray-box models: partially known

Correlation analysis is truly a nonparametric method; it produces an *impulse response model*.

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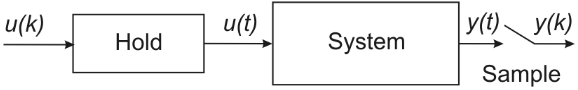
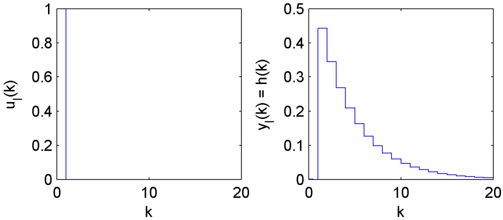
Recall: discrete-time model



Discrete-time impulse response



Discrete-time, unit impulse signal:

$$u_{\mathrm{I}}(k) = \begin{cases} 1 & k = 0 \\ 0 & k > 0 \end{cases}$$

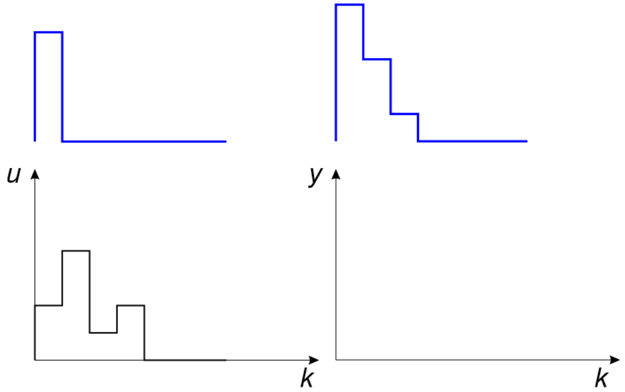
(does not have area 1, so it's different from the discrete-time realization of the continuous-time impulse!)

Discrete-time impulse response:

$$y_I(k) = h(k), \quad k \geq 0$$

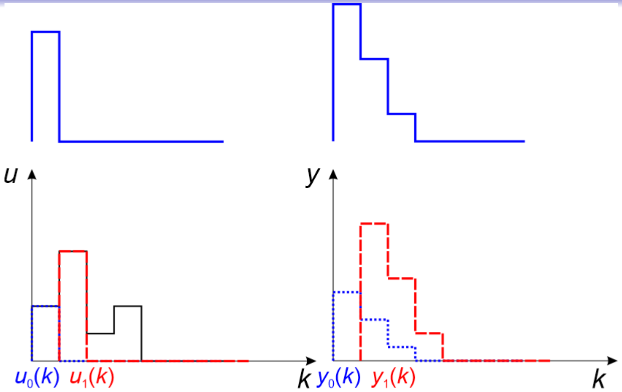
$h(k), k \geq 0$ is also called the weighting function of the system.

Impulse response model: Problem



Take a discrete-time input $u(k)$. Our objective is to find the resulting output $y(k)$.

Impulse response model: Input decomposition



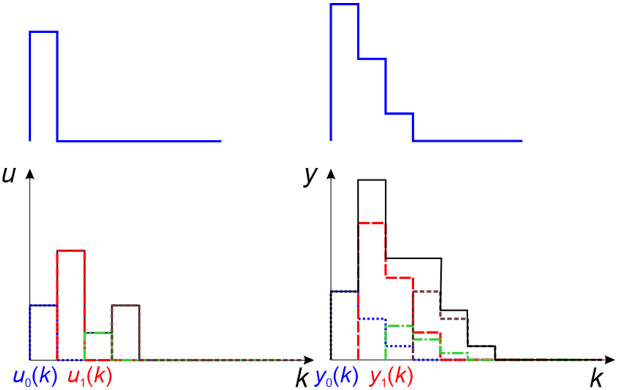
Consider a signal $\tilde{u}_j(k)$ equal to $u(j)$ at $k = j$, and 0 elsewhere; just a shifted and scaled unit impulse:

$$\tilde{u}_j(k) = u(j)u_{\mathrm{I}}(k - j)$$

So, the response to $\tilde{u}_j(k)$ is a shifted and scaled impulse response:

$$\tilde{y}_j(k) = u(j)h(k - j)$$

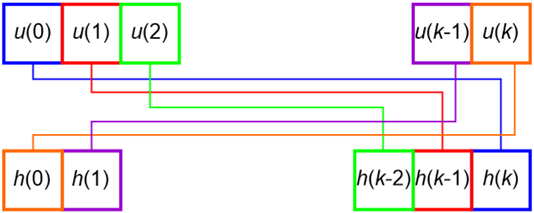
Impulse response model: Superposition



Now, $u(k)$ is the superposition of all signals $\tilde{u}_j$, so due to linearity:

$$y(k) = \sum_{j=0}^{k} \tilde{y}_j(k) = \sum_{j=0}^{k} u(j)h(k-j)$$

Impulse response model: Convolution



$$y(k) = \sum_{j=0}^{k} \tilde{y}_j(k) = \sum_{j=0}^{k} u(j)h(k-j) = \sum_{j=0}^{k} h(j)u(k-j) = \sum_{j=0}^{\infty} h(j)u(k-j)$$

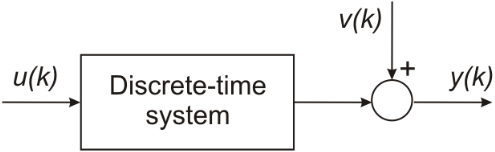
where zero initial conditions were assumed, i.e. $u(j) = 0 \forall j < 0$.

Impulse-response model

The response to an arbitrary signal $u(k)$ is the *convolution* of the input and the impulse response:

$$y(k) = \sum_{j=0}^{\infty} h(j)u(k-j) + v(k)$$

where we included an additional disturbance term $v(k)$.



Assumptions

Assumptions

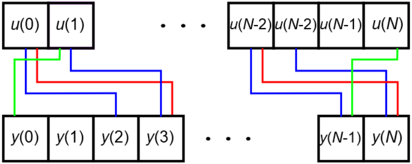
1. The input $u(k)$ is a stationary stochastic process.
2. The input $u(k)$ and the disturbance $v(k)$ are independent.

Recall:

- Independence of random variables.
- Stationary stochastic process: constant mean at every time step, covariance only depends on difference between time steps and not on absolute time.

Covariance function between input and output

$$r_{yu}(\tau) = \mathrm{E}\left\{y(k+\tau)u(k)\right\} \quad \left(= \frac{1}{\#}\sum_k y(k+\tau)u(k)\right)$$



For instance:

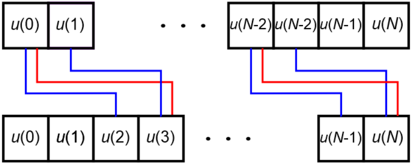
$$r_{yu}(2) = \mathrm{E}\left\{y(k+2)u(k)\right\} = \frac{1}{\#}\sum_k y(k+2)u(k)$$

$$r_{yu}(3) = \mathrm{E}\left\{y(k+3)u(k)\right\}$$

$$r_{yu}(-1) = \mathrm{E}\left\{y(k-1)u(k)\right\} = \mathrm{E}\left\{y(k)u(k+1)\right\}$$

Covariance function of the input

$$r_u(\tau) = r_u(-\tau) = \mathrm{E}\left\{u(k+\tau)u(k)\right\} \left(= \frac{1}{\#}\sum_k u(k+\tau)u(k) \right)$$



For instance:

$$r_u(2) = r_u(-2) = \mathrm{E}\left\{u(k+2)u(k)\right\}$$
$$r_u(3) = r_u(-3) = \mathrm{E}\left\{u(k+3)u(k)\right\}$$

Notes: r_u is symmetrical, and r_{yu}, r_u are true covariances only when the input and output are zero-mean, so if they are not, then the means must be subtracted prior to applying the method.

Relationship of covariances and impulse response

If there were no disturbance, then:

$$\begin{aligned} r_{yu}(\tau) &= \mathrm{E}\left\{y(k+\tau)u(k)\right\} \\ &= \mathrm{E}\left\{\left[\sum_{j=0}^{\infty} h(j)u(k+\tau-j)\right]u(k)\right\} \\ &= \sum_{j=0}^{\infty} h(j)\mathrm{E}\left\{u(k+\tau-j)u(k)\right\} = \sum_{j=0}^{\infty} h(j)r_u(\tau-j) \end{aligned}$$

The errors coming from the disturbance are dealt with later, implicitly, using linear regression.

Impulse response identification

Writing the covariance relationship for all τ:

$$r_{yu}(0) = \sum_{j=0}^{\infty} h(j) r_u(-j) = h(0) r_u(0) + h(1) r_u(-1) + h(2) r_u(-2) + \dots$$

$$r_{yu}(1) = \sum_{j=0}^{\infty} h(j) r_u(1-j) = h(0) r_u(1) + h(1) r_u(0) + h(2) r_u(-1) + \dots$$

$$\dots$$

we obtain (in principle) an infinite system of linear equations:

- Coefficients $r_u(\tau)$, $r_{yu}(\tau)$.
- Unknowns $h(0), h(1), \dots$: solution of the system.

Linear system structure

$$r_{yu}(\tau) = \sum_{j=0}^{\infty} h(j) r_u(-j) = h(0) r_u(\tau) + h(1) r_u(\tau - 1) + h(2) r_u(\tau - 2) + \dots$$

$$\begin{array}{|c|c|c|c|} \hline r_u(0) & r_u(1) & r_u(2) & r_u(3) \\ \hline \end{array} \quad \cdots$$

$$\begin{bmatrix} r_{yu}(0) \\ r_{yu}(1) \\ r_{yu}(2) \\ r_{yu}(3) \\ \vdots \end{bmatrix} = \begin{bmatrix} r_u(0) & r_u(1) & r_u(2) & r_u(3) & \cdots & & & \\ r_u(1) & r_u(0) & r_u(1) & r_u(2) & r_u(3) & \cdots & & \\ r_u(2) & r_u(1) & r_u(0) & r_u(1) & r_u(2) & r_u(3) & \cdots & \\ r_u(3) & r_u(2) & r_u(1) & r_u(0) & r_u(1) & r_u(2) & r_u(3) & \cdots \\ \vdots & & & & & & & \end{bmatrix} \cdot \begin{bmatrix} h(0) \\ h(1) \\ h(2) \\ h(3) \\ \vdots \end{bmatrix}$$

Next, a practical algorithm working with finite data is given.

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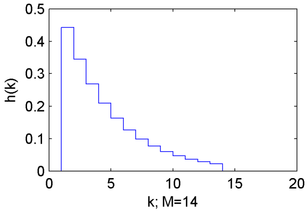
Finite impulse response model



Impose the condition $h(k) = 0$ for $k \geq M$. We obtain the finite impulse response (FIR) model:

$$y(k) = \sum_{j=0}^{M-1} h(j)u(k-j) + v(k)$$

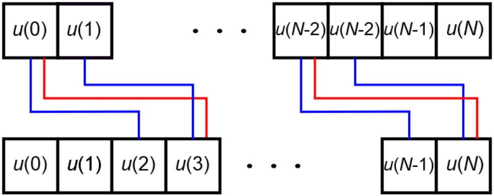
Note: M must be taken so that $MT_s \gg$ dominant time constants (or equivalently, the system is close to steady-state)

Covariances from data: r_u

Consider we are given signals $u(k), y(k)$ with $k = 0, \ldots, N$.
We have, for positive τ:

$$\begin{aligned} r_u(\tau) &= \mathrm{E}\{u(k+\tau)u(k)\} \\ &\approx \boxed{\frac{1}{N}\sum_{k=0}^{N-\tau} u(k+\tau)u(k)} \\ &=: \widehat{r}_u(\tau), \qquad \forall \tau \geq 0 \end{aligned}$$

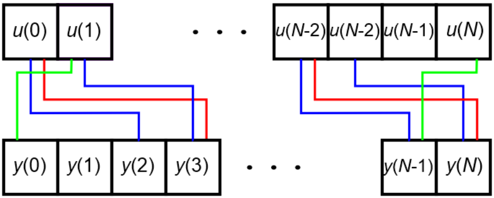
and $\widehat{r}_u(-\tau) = \widehat{r}_u(\tau)$ due to symmetry.



Covariances from data: r_{yu}

For positive and negative τ:

$$r_{yu}(\tau) = \mathrm{E}\left\{y(k+\tau)u(k)\right\}$$
$$\approx \begin{cases} \dfrac{1}{N}\displaystyle\sum_{k=0}^{N-\tau} y(k+\tau)u(k) & \text{if } \tau \geq 0 \\ \dfrac{1}{N}\displaystyle\sum_{k=-\tau}^{N} y(k+\tau)u(k) & \text{if } \tau < 0 \end{cases}$$
$$=: \widehat{r}_{yu}(\tau), \qquad \forall \tau \geq 0$$



Finite covariance relationship

FIR equation:

$$y(k) = \sum_{j=0}^{M-1} h(j)u(k-j) + v(k)$$

The covariance relationship is similarly truncated:

$$r_{yu}(\tau) = \sum_{j=0}^{M-1} h(j)r_u(\tau - j)$$

Linear system

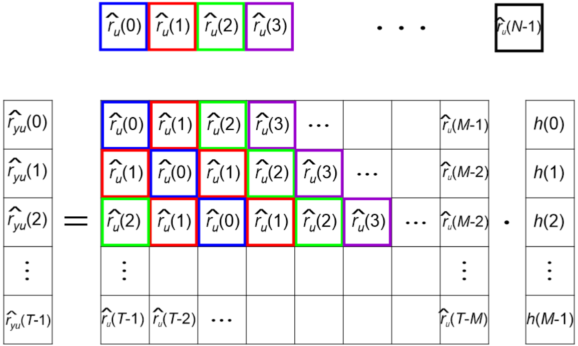
Using $\widehat{r}_{yu}$, $\widehat{r}_u$ estimated from data, write the truncated equations for $\tau = 0, \ldots, T-1$ (keeping in mind that $\widehat{r}_u(-\tau) = \widehat{r}_u(\tau)$):

$$
\begin{aligned}
\widehat{r}_{yu}(0) &= \sum_{j=0}^{M-1} h(j)\widehat{r}_u(-j) \\
&= h(0)\widehat{r}_u(0) + h(1)\widehat{r}_u(1) + \ldots + h(M-1)\widehat{r}_u(M-1) \\
\widehat{r}_{yu}(1) &= \sum_{j=0}^{M-1} h(j)\widehat{r}_u(1-j) \\
&= h(0)\widehat{r}_u(1) + h(1)\widehat{r}_u(0) + \ldots + h(M-1)\widehat{r}_u(M-2) \\
\ldots \\
\widehat{r}_{yu}(T-1) &= \sum_{j=0}^{M-1} h(j)\widehat{r}_u(T-1-j) \\
&= h(0)\widehat{r}_u(T-1) + h(1)\widehat{r}_u(T-2) + \ldots + h(M-1)\widehat{r}_u(T-M)
\end{aligned}
$$

– a linear system of T equations in M unknowns $h(0), \ldots, h(M-1)$.

Linear system: Matrix form

$$\begin{bmatrix} \hat{r}_{yu}(0) \\ \hat{r}_{yu}(1) \\ \vdots \\ \hat{r}_{yu}(T-1) \end{bmatrix} = \begin{bmatrix} \hat{r}_u(0) & \hat{r}_u(1) & \dots & \hat{r}_u(M-1) \\ \hat{r}_u(1) & \hat{r}_u(0) & \dots & \hat{r}_u(M-2) \\ \vdots & & & \\ \hat{r}_u(T-1) & \hat{r}_u(T-2) & \dots & \hat{r}_u(T-M) \end{bmatrix} \cdot \begin{bmatrix} h(0) \\ h(1) \\ \vdots \\ h(M-1) \end{bmatrix}$$



Linear system: Notes

Naively taking $T = M$ would give an exact system solution, but due to noise and disturbances this solution would be overfitted. So it is necessary to take $T > M$ (preferably, $T \gg M$).

Then we can apply the machinery of linear regression (see Part 3) to solve this problem.

Using the FIR model

Once the system has been solved for the estimated $\hat{h}$, we predict outputs with:

$$\hat{y}(k) = \sum_{j=0}^{M-1} \hat{h}(j)u(k-j)$$

Special case: White noise input

Consider the case when the input $u(k)$ is zero-mean white noise.

Then, $r_u(\tau) = 0$ whenever $\tau \neq 0$ (since white noise is uncorrelated), and $r_{yu}(\tau) = \sum_{j=0}^{\infty} h(j) r_u(\tau - j)$ simplifies to:

$$r_{yu}(\tau) = h(\tau) r_u(0)$$

This leads to the easy algorithm:

$$\hat{h}(\tau) = \frac{\widehat{r}_{yu}(\tau)}{\widehat{r}_u(0)}$$

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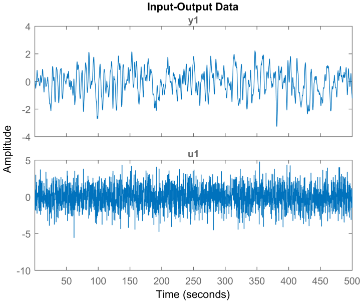
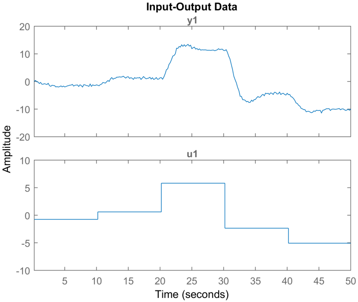
Experimental data

Consider we are given the following, separate, identification and validation data sets.

`plot(id);` and `plot(val);`

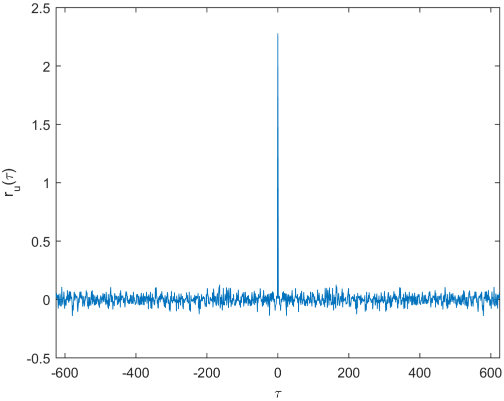




There are 2500 samples in the identification data. We notice that the data is zero-mean.

Input covariance

`[c, tau] = xcorr(id.u);` and `plot(tau, c);`



The input is white noise.

Applying correlation analysis

`fir = cra(id, M, 0);` or `fir = cra(id, M, 0, plotlevel);`

Arguments:

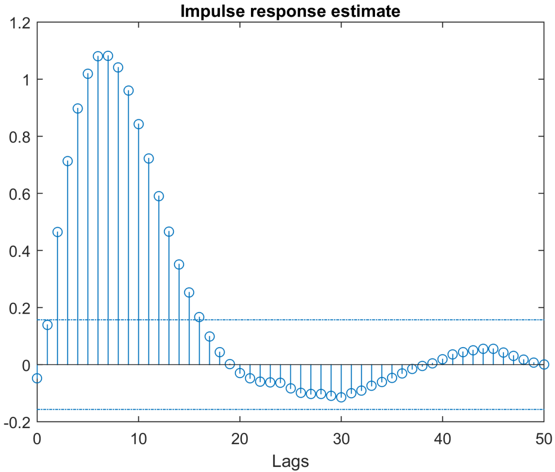
1. Identification data.
2. FIR length M, here it is set to 50.
3. Third argument 0 means no *input whitening* is performed.

Dealing with non-ideal inputs:

- If input is not zero-mean, pass the data through `detrend` to remove the means.
- If input is not white noise, the third argument should be left to default (by not specifying it or setting it to an empty matrix), which means input whitening is performed.

Applying correlation analysis (continued)

By default (or with `plotlevel=1`) the FIR parameters are shown with a 99% confidence interval.

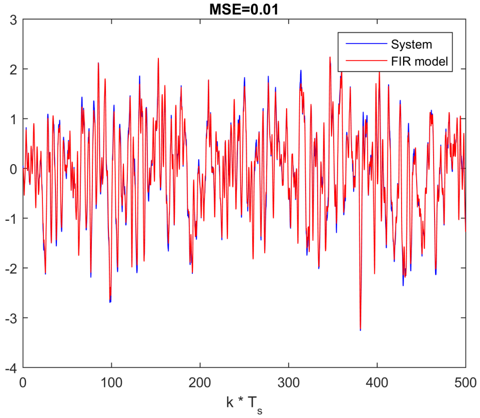


`plotlevel=2` also produces the covariance functions.

Results on the identification data

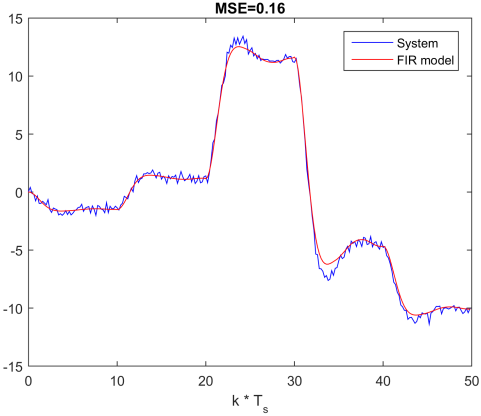
```
yhat = conv(fir, id.u); yhat = yhat(1:length(id.u));
```

To simulate the FIR model, a *convolution* between the FIR parameters and the input is performed. The simulated output is longer than needed so we cut it off at the right length.



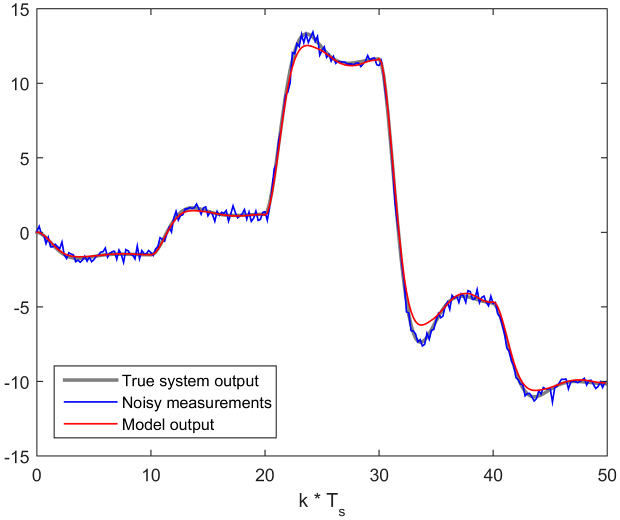
Validation of the FIR model

```
yhat = conv(fir, val.u); yhat = yhat(1:length(val.u));
```



Results OK, not great.

Insight into the different signals



Alternative: `impulseest` function

`model = impulseest(id, M);` or `model = impulseest(id);`

Uses a more involved algorithm than the one studied in the lectures.

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Simplified guarantee in the white-noise case

Additional assumption

3. The input $u(k)$ is zero-mean white noise.

Theorem

In the white-noise case, as the number of data points N grows to infinity, the estimates $\widehat{h}(\tau)$ converge to the true values $h(\tau)$.

Remark: This type of property, where the true solution is obtained in the limit of infinite data, is called *consistency*.

Summary

- Discrete-time unit impulse and impulse response h.
- Using impulse response as a model: convolution with input u.
- Ideal covariance functions and linear system of equations in h.
- Practical correlation analysis:
    - covariance from finite data
    - finite impulse response (FIR) model
    - finite-dimensional linear system
- Matlab example.
- Simplified accuracy guarantee (consistency for infinite data).