System Identification

Control Engineering EN, 3rd year B.Sc.
Technical University of Cluj-Napoca
Romania

Lecturer: Lucian Buşoniu

Intro & motivation
000000

Recursive least-squares and ARX
0000000000000000000

Recursive instrumental variables
00000000

Part IX

Recursive identification methods

Intro & motivation
○○○○○○

Recursive least-squares and ARX
○○○○○○○○○○○○○○○○○○○

Recursive instrumental variables
○○○○○○○○

Table of contents

1. Introduction and motivation

2. Recursive least-squares and ARX

3. Recursive instrumental variables

Recursive identification: Idea
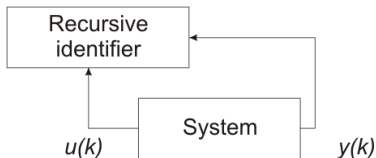
Recursive methods can work *online*, while the system is running.

At step k, compute *new parameter estimate* $\widehat{\theta}(k)$, based on the previous estimate $\widehat{\theta}(k-1)$ and newly available data $u(k)$, $y(k)$.

Remark: To contrast them with recursive identification, the previous methods, which used the whole data set at once, will be called *batch* identification.

Motivation

Recursive methods:

- Require less memory, and less computation *for each update*, than the whole batch algorithm.
- ⇒ Easier to apply in real-time.
  (total computation, after the entire dataset, may be larger)
- When properly modified, can deal with systems that change over time (*time-varying*).
- If model is used to tune a controller, we obtain *adaptive control*:

Intro & motivation
○○●○○○

Recursive least-squares and ARX
○○○○○○○○○○○○○○○○○○

Recursive instrumental variables
○○○○○○○○
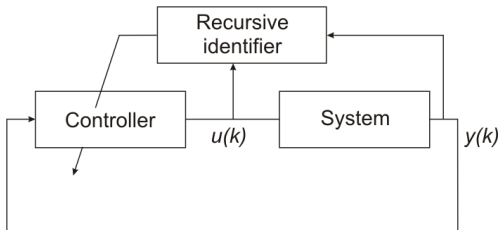
Disadvantage

Recursive methods are usually approximations of batch techniques
$\Rightarrow$ guarantees more difficult to obtain.

Intro & motivation
○○○●○○

Recursive least-squares and ARX
○○○○○○○○○○○○○○○○○○○

Recursive instrumental variables
○○○○○○○○

Motivating example: Estimating a scalar

Recall scalar estimation. Model:

$$y(k) = b + e(k) = 1 \cdot b + e(k) = \varphi(k)\theta + e(k)$$

where $\varphi(k) = 1 \forall k$, $\theta = b$.

For the data points up to and including k:

$$y(1) = \varphi(1)\theta = 1 \cdot b$$
$$\cdots$$
$$y(k) = \varphi(k)\theta = 1 \cdot b$$

After calculation, the solution of this system is:

$$\widehat{\theta}(k) = \frac{1}{k}[y(1) + \ldots + y(k)]$$

(estimate is the average of all measurements, filtering out the noise).

Intro & motivation
○○○○●○

Recursive least-squares and ARX
○○○○○○○○○○○○○○○○○

Recursive instrumental variables
○○○○○○○○

Estimating a scalar: Recursive formulation

Rewrite the formula:

$$\widehat{\theta}(k) = \frac{1}{k}[y(1) + \ldots + y(k)]$$
$$= \frac{1}{k}[(k-1)\frac{1}{k-1}(y(1) + \ldots + y(k-1)) + y(k)]$$
$$= \frac{1}{k}[(k-1)\widehat{\theta}(k-1) + y(k)]$$

(already a recursive formula, but we go on to gain more intuition)

$$= \frac{1}{k}[k\widehat{\theta}(k-1) + y(k) - \widehat{\theta}(k-1)]$$
$$= \widehat{\theta}(k-1) + \frac{1}{k}[y(k) - \widehat{\theta}(k-1)]$$

Intro & motivation
○○○○○●

Recursive least-squares and ARX
○○○○○○○○○○○○○○○○○○○○

Recursive instrumental variables
○○○○○○○○

Estimating a scalar: Features

$$\widehat{\theta}(k) = \widehat{\theta}(k-1) + \frac{1}{k}[y(k) - \widehat{\theta}(k-1)]$$

Method has many of the features of general recursive techniques:

- Recursive formula: new estimate $\widehat{\theta}(k)$ computed based on previous estimate $\widehat{\theta}(k-1)$ and new data $y(k)$.
- $[y(k) - \widehat{\theta}(k-1)]$ is a prediction error $\varepsilon(k)$, since $\widehat{\theta}(k-1) = \widehat{b} = \widehat{y}(k)$, a one-step-ahead prediction of the output.
- Update applies a correction proportional to $\varepsilon(k)$, weighted by $\frac{1}{k}$:
  - When the error $\varepsilon(k)$ is large (or small), a large (or small) adjustment is made.
  - The weight $\frac{1}{k}$ decreases with time k, so adjustments get smaller as $\widehat{\theta}$ becomes better.

Intro & motivation
000000

Recursive least-squares and ARX
0000000000000000000

Recursive instrumental variables
00000000

Table of contents

Recall: Least-squares regression

Model:

$$y(k) = \varphi^\top(k)\theta + e(k)$$

Dataset up to *k* gives a linear system of equations:

$$y(1) = \varphi^\top(1)\theta$$
$$y(2) = \varphi^\top(2)\theta$$
$$\cdots$$
$$y(k) = \varphi^\top(k)\theta$$

After some linear algebra and calculus, the least-squares solution can be written:

$$\widehat{\theta}(k) = \left[\sum_{j=1}^{k} \varphi(j)\varphi^\top(j)\right]^{-1} \left[\sum_{j=1}^{k} \varphi(j)y(j)\right]$$

Intro & motivation
oooooo

Recursive least-squares and ARX
o●oooooooooooooooooo

Recursive instrumental variables
ooooooo

Least-squares: Recursive formula

With notation $P(k) = \sum_{j=1}^{k} \varphi(j)\varphi^\top(j)$:

$$
\begin{aligned}
\widehat{\theta}(k) &= P^{-1}(k) \left[\sum_{j=1}^{k} \varphi(j)y(j) \right] \\
&= P^{-1}(k) \left[\sum_{j=1}^{k-1} \varphi(j)y(j) + \varphi(k)y(k) \right] \\
&= P^{-1}(k) \left[P(k-1)\widehat{\theta}(k-1) + \varphi(k)y(k) \right] \\
&= P^{-1}(k) \left[[P(k) - \varphi(k)\varphi^\top(k)]\widehat{\theta}(k-1) + \varphi(k)y(k) \right] \\
&= \widehat{\theta}(k-1) + P^{-1}(k) \left[-\varphi(k)\varphi^\top(k)\widehat{\theta}(k-1) + \varphi(k)y(k) \right] \\
&= \widehat{\theta}(k-1) + P^{-1}(k)\varphi(k) \left[y(k) - \varphi^\top(k)\widehat{\theta}(k-1) \right]
\end{aligned}
$$

Least-squares: Features

$$\boxed{\widehat{\theta}(k) = \widehat{\theta}(k-1) + P^{-1}(k)\varphi(k)\left[y(k) - \varphi^{\top}(k)\widehat{\theta}(k-1)\right]}$$

$$\widehat{\theta}(k) = \widehat{\theta}(k-1) + W(k)\varepsilon(k)$$

- Recursive formula: new estimate $\widehat{\theta}(k)$ computed based on previous estimate $\widehat{\theta}(k-1)$ and new data $y(k)$.
- $\left[y(k) - \varphi^{\top}(k)\widehat{\theta}(k-1)\right]$ is a prediction error $\varepsilon(k)$, since $\varphi^{\top}(k)\widehat{\theta}(k-1) = \widehat{y}(k)$ is the one-step-ahead prediction using the previous parameter vector.
- $W(k) = P^{-1}(k)\varphi(k)$ is a weighting vector: elements of P grow large for large k, therefore $W(k)$ decreases.

Intro & motivation
000000

Recursive least-squares and ARX
000●000000000000000

Recursive instrumental variables
00000000

Recursive matrix inversion

The previous formula requires matrix inverse $P^{-1}(k)$, which is computationally costly.

For P, an easy recursion exists: $P(k) = P(k-1) + \varphi(k)\varphi^\top(k)$, but this does not help; the matrix must still be inverted.

The Sherman-Morrison formula gives a recursion for the inverse P^{-1}:

$$P^{-1}(k) = P^{-1}(k-1) - \frac{P^{-1}(k-1)\varphi(k)\varphi^\top(k)P^{-1}(k-1)}{1 + \varphi^\top(k)P^{-1}(k-1)\varphi(k)}$$

Exercise: Prove the Sherman-Morrison formula! (linear algebra)

Intro & motivation
000000

Recursive least-squares and ARX
0000●00000000000000

Recursive instrumental variables
00000000

Overall algorithm

Recursive least-squares (RLS)

initialize $\widehat{\theta}(0)$, $P^{-1}(0)$

loop at every step $k = 1, 2, \ldots$

  measure $y(k)$, form regressor vector $\varphi(k)$

  find prediction error $\varepsilon(k) = y(k) - \varphi^\top(k)\widehat{\theta}(k-1)$

  update inverse $P^{-1}(k) = P^{-1}(k-1) - \frac{P^{-1}(k-1)\varphi(k)\varphi^\top(k)P^{-1}(k-1)}{1+\varphi^\top(k)P^{-1}(k-1)\varphi(k)}$

  compute weights $W(k) = P^{-1}(k)\varphi(k)$

  update parameters $\widehat{\theta}(k) = \widehat{\theta}(k-1) + W(k)\varepsilon(k)$

end loop

Initialization

The RLS algorithm requires initial parameter $\widehat{\theta}(0)$, and initial inverse $P^{-1}(0)$.

Typical choices without prior knowledge:

- $\widehat{\theta}(0) = [0, \ldots, 0]^\top$ - a vector of n zeros.
- $P^{-1}(0) = \frac{1}{\delta} I$, with δ a small number such as 10^{-3}.
  An equivalent initial value of P would be $P(0) = \delta I$.

Intuition: P small corresponds to a low confidence in the initial parameters. This leads to large $P^{-1}(0)$, so initially the weights $W(k)$ are large and large updates are applied to $\widehat{\theta}$.

If a prior value $\widehat{\theta}$ is available, $\widehat{\theta}(0)$ can be initialized to it, and δ correspondingly increased. This means better confidence in the initial estimate, and smaller initial updates.

Intro & motivation
000000

Recursive least-squares and ARX
0000000●0000000000000

Recursive instrumental variables
00000000

Sanity check: Recovering the scalar case

$$y(k) = b + e(k) = \varphi(k)\theta + e(k), \text{ where } \varphi(k) = 1, \theta = b$$

Take $\widehat{\theta}(0) = 0$, $P^{-1}(0) \to \infty$. We have with Sherman-Morrison:

$$P^{-1}(k) = P^{-1}(k-1) - \frac{(P^{-1}(k-1))^2}{1 + P^{-1}(k-1)} = \frac{P^{-1}(k-1)}{1 + P^{-1}(k-1)}$$

$$P^{-1}(1) = 1$$

$$P^{-1}(2) = \frac{1}{2}$$

$$\cdots$$

$$P^{-1}(k) = \frac{1}{k}$$

Also, $\varepsilon(k) = y(k) - \widehat{\theta}(k-1)$ and $W(k) = P^{-1}(k) = \frac{1}{k}$, leading to:

$$\widehat{\theta}(k) = \widehat{\theta}(k-1) + W(k)\varepsilon(k) = \widehat{\theta}(k-1) + \frac{1}{k}[y(k) - \widehat{\theta}(k-1)]$$
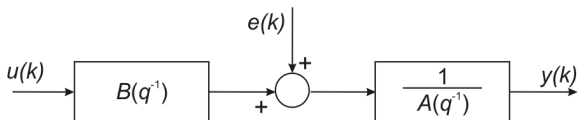
$\Rightarrow$ formula for the scalar case was recovered.

Intro & motivation
○○○○○○

Recursive least-squares and ARX
○○○○○○○●○○○○○○○○○○

Recursive instrumental variables
○○○○○○○○

Table of contents

Intro & motivation
000000

Recursive least-squares and ARX
00000000●0000000000

Recursive instrumental variables
00000000

Recall: ARX model

$$A(q^{-1})y(k) = B(q^{-1})u(k) + e(k)$$
$$(1 + a_1 q^{-1} + \cdots + a_{na} q^{-na})y(k) =$$
$$(b_1 q^{-1} + \cdots + b_{nb} q^{-nb})u(k) + e(k)$$

In explicit form:

$$y(k) + a_1 y(k-1) + a_2 y(k-2) + \ldots + a_{na} y(k-na)$$
$$= b_1 u(k-1) + b_2 u(k-2) + \ldots + b_{nb} u(k-nb) + e(k)$$

where the model parameters are: $a_1, a_2, \ldots, a_{n_a}$ and $b_1, b_2, \ldots, b_{nb}$.

Intro & motivation
oooooo

Recursive least-squares and ARX
ooooooooo**oo**●**oo**oooooooo

Recursive instrumental variables
oooooooo

Recall: Linear regression representation

$$
\begin{aligned}
y(k) = &- a_1 y(k-1) - a_2 y(k-2) - \ldots - a_{na} y(k-na) \\
&b_1 u(k-1) + b_2 u(k-2) + \ldots + b_{nb} u(k-nb) + e(k) \\
= &\begin{bmatrix} -y(k-1) & \cdots & -y(k-na) & u(k-1) & \cdots & u(k-nb) \end{bmatrix} \\
&\cdot \begin{bmatrix} a_1 & \cdots & a_{na} & b_1 & \cdots & b_{nb} \end{bmatrix}^\top + e(k) \\
=: &\varphi^\top(k)\theta + e(k)
\end{aligned}
$$

Regressor vector: $\varphi \in \mathbb{R}^{na+nb}$, previous output and input values.

Parameter vector: $\theta \in \mathbb{R}^{na+nb}$, polynomial coefficients.

Recursive ARX

With this representation, just an instantiation of RLS template.

Recursive ARX

initialize $\widehat{\theta}(0)$, $P^{-1}(0)$
loop at every step $k = 1, 2, \ldots$
    measure $u(k)$, $y(k)$
    form regressor vector
    $\varphi(k) = [-y(k-1), \cdots, -y(k-na), u(k-1), \cdots, u(k-nb)]^\top$
    find prediction error $\varepsilon(k) = y(k) - \varphi^\top(k)\widehat{\theta}(k-1)$
    update inverse $P^{-1}(k) = P^{-1}(k-1) - \frac{P^{-1}(k-1)\varphi(k)\varphi^\top(k)P^{-1}(k-1)}{1+\varphi^\top(k)P^{-1}(k-1)\varphi(k)}$
    compute weights $W(k) = P^{-1}(k)\varphi(k)$
    update parameters $\widehat{\theta}(k) = \widehat{\theta}(k-1) + W(k)\varepsilon(k)$
end loop

Remark: Outputs more than *na* steps ago, and inputs more than *nb* steps ago, are not used so they can be forgotten, reducing memory usage.

Intro & motivation
000000

Recursive least-squares and ARX
0000000000000●0000000

Recursive instrumental variables
00000000

Guarantees idea

With $\widehat{\theta}(0) = 0$, $P^{-1}(0) = \frac{1}{\delta}I$ and $\delta \to 0$, recursive ARX at step k is equivalent to running batch ARX on the dataset $u(1), y(1), \ldots, u(k), y(k)$.

$\Rightarrow$ the same guarantees as batch ARX:

Theorem

Under appropriate assumptions (including the existence of true parameters θ_0), ARX identification is consistent: the estimated parameters $\widehat{\theta}$ tend to the true parameters θ_0 as $k \to \infty$.

The condition on P is to ensure the inverses are the same as in the offline case.

Table of contents

Intro & motivation
000000

Recursive least-squares and ARX
0000000000000●000000

Recursive instrumental variables
00000000

System

To illustrate recursive ARX, we take a known system:

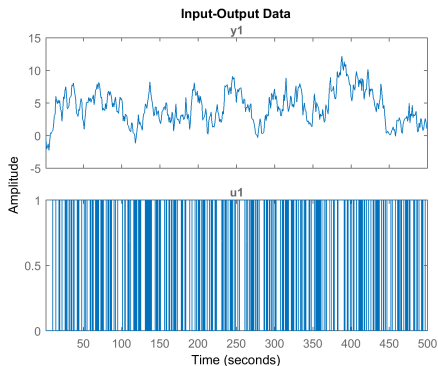$$y(k) + ay(k-1) = bu(k-1) + e(k), \quad a = -0.9, b = 1$$

(Söderström & Stoica)

```
system = idpoly([1 -0.9], [0 1]);
```

Identification data obtained in simulation:

```
sim(system, u, 'noise');
```

Intro & motivation
000000

Recursive least-squares and ARX
0000000000000**0**0000

Recursive instrumental variables
00000000

Recursive ARX

```
model = rarx(id, [na, nb, nk], 'ff', 1, th0, Pinv0);
```

Arguments:

1. Identification data.
2. Array containing the orders of *A* and *B* and the delay *nk*.
3. `'ff'`, `1` selects the algorithm variant presented in this lecture.
4. `th0` is the initial parameter value.
5. `Pinv0` is the initial inverse $P^{-1}(0)$.

Intro & motivation
000000

Recursive least-squares and ARX
0000000000000●000

Recursive instrumental variables
00000000

Results

$na = nb = nk = 1$
$P^{-1}(0) = \frac{1}{\delta} I$, left: $\delta = 10^{-3}$, right: $\delta = 100$.
For all the experiments, $\theta_0 = [0, 0]^\top$.
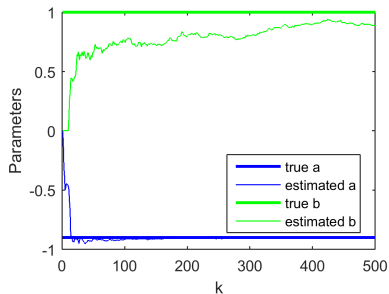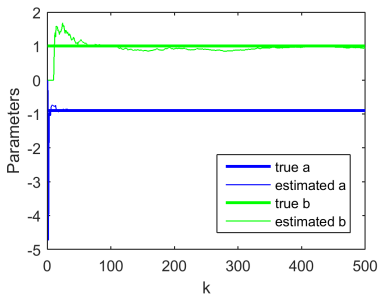
Conclusions:

- Convergence to true parameter values...
- ...slower when δ is larger (in this case, misplaced large confidence in the initial solution)

Intro & motivation
oooooo

Recursive least-squares and ARX
oooooooooooooo○○○○●○○

Recursive instrumental variables
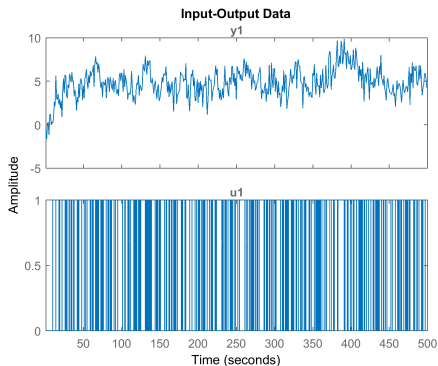oooooooo

System outside model class

Take now an *output-error* system (so it is not ARX with white noise):

$$y(k) = \frac{bq^{-1}}{1 + fq^{-1}} u(k) + e(k), \quad f = -0.9, b = 1$$

(Söderström & Stoica)

```
system = idpoly([], [0 1], [], [], [1 -.9]);
```

Identification data:

Intro & motivation
000000

Recursive least-squares and ARX
0000000000000000000●0

Recursive instrumental variables
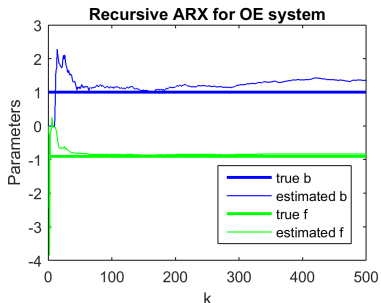00000000

Results

$nf = nb = nk = 1$, so we attempt the ARX model:

$$y(k) + fy(k-1) = bu(k-1) + e(k)$$

Also, $P^{-1}(0) = \frac{1}{\delta}I$, $\delta = 10^{-3}$.

Conclusion: Convergence to the true values is not attained! – due to colored noise (system not in model class)

Other recursive PEM algorithms

Other recursive PEM methods available in Matlab, e.g.:

- ARMAX, `rarmax`
- output-error, `roe`
- generic prediction error method, `rpem`

Table of contents

Recall: Instrumental variable method

IV methods find the parameter vector using:

$$\widehat{\theta} = \left[\frac{1}{N} \sum_{k=1}^{N} Z(k)\varphi^{\top}(k) \right]^{-1} \left[\frac{1}{N} \sum_{k=1}^{N} Z(k)y(k) \right]$$

which is the solution to the system of equations:

$$\left[\frac{1}{N} \sum_{k=1}^{N} Z(k)[\varphi^{\top}(k)\theta - y(k)] \right] = 0$$

Regressor vector:
$\varphi(k) = [-y(k-1), \cdots, -y(k-na), u(k-1), \cdots, u(k-nb)]^{\top}$,
instrument vector usually:
$Z(k) = [-x(k-1), \cdots, -x(k-na), u(k-1), \cdots, u(k-nb)]^{\top}$.

Since instrument vector $Z(k)$ uncorrelated with the noise,
IV methods can handle colored noise.

Recursive formulation (1)

Averaging by the number of data points can be removed, leading to:

$$\widehat{\theta} = \left[\sum_{k=1}^{N} Z(k)\varphi^{\top}(k)\right]^{-1} \left[\sum_{k=1}^{N} Z(k)y(k)\right]$$

Averaging was needed in batch IV since summations over many data points could grow very large, leading to numerical problems.
Recursive IV will, in the end, add terms one by one, and will therefore be numerically more stable.

Rewrite the equation for the recursive case:

$$\widehat{\theta}(k) = P^{-1}(k) \left[\sum_{j=1}^{k} Z(j)y(j)\right]$$

where $P(k) = \sum_{j=1}^{k} Z(j)\varphi^{\top}(j)$.

Intro & motivation
oooooo

Recursive least-squares and ARX
oooooooooooooooooooo

Recursive instrumental variables
oo●ooooo

Recursive formulation (2)

With this definition of P, the recursive update is found similarly to RLS:

$$
\begin{aligned}
\widehat{\theta}(k) &= P^{-1}(k) \left[\sum_{j=1}^{k} Z(j)y(j) \right] \\
&= P^{-1}(k) \left[\sum_{j=1}^{k-1} Z(j)y(j) + Z(k)y(k) \right] \\
&= P^{-1}(k) \left[P(k-1)\widehat{\theta}(k-1) + Z(k)y(k) \right] \\
&= P^{-1}(k) \left[[P(k) - Z(k)\varphi^{\top}(k)]\widehat{\theta}(k-1) + Z(k)y(k) \right] \\
&= \widehat{\theta}(k-1) + P^{-1}(k) \left[-Z(k)\varphi^{\top}(k)\widehat{\theta}(k-1) + Z(k)y(k) \right] \\
&= \widehat{\theta}(k-1) + P^{-1}(k)Z(k) \left[y(k) - \varphi^{\top}(k)\widehat{\theta}(k-1) \right]
\end{aligned}
$$

Intro & motivation
oooooo

Recursive least-squares and ARX
ooooooooooooooooooooo

Recursive instrumental variables
oooeoooo

Recursive formulation (3)

Final formula:

$$\boxed{\widehat{\theta}(k) = \widehat{\theta}(k-1) + P^{-1}(k)Z(k)\left[y(k) - \varphi^\top(k)\widehat{\theta}(k-1)\right]}$$

$$\widehat{\theta}(k) = \widehat{\theta}(k-1) + W(k)\varepsilon(k)$$

with the Sherman-Morrison update of the inverse:

$$P^{-1}(k) = P^{-1}(k-1) - \frac{P^{-1}(k-1)Z(k)\varphi^\top(k)P^{-1}(k-1)}{1 + \varphi^\top(k)P^{-1}(k-1)Z(k)}$$

Intro & motivation
oooooo

Recursive least-squares and ARX
oooooooooooooooooooo

Recursive instrumental variables
ooooo●ooo

Recursive IV: Overall algorithm

Recursive IV

initialize $\widehat{\theta}(0)$, $P^{-1}(0)$
loop at every step $k = 1, 2, \ldots$
  measure $y(k)$, form regressor vector $\varphi(k)$ and instrument vector
  $Z(k) = [-x(k-1), \cdots, -x(k-na), u(k-1), \cdots, u(k-nb)]^\top$
  find prediction error $\varepsilon(k) = y(k) - \varphi^\top(k)\widehat{\theta}(k-1)$
  update inverse $P^{-1}(k) = P^{-1}(k-1) - \dfrac{P^{-1}(k-1)Z(k)\varphi^\top(k)P^{-1}(k-1)}{1+\varphi^\top(k)P^{-1}(k-1)Z(k)}$
  compute weights $W(k) = P^{-1}(k)Z(k)$
  update parameters $\widehat{\theta}(k) = \widehat{\theta}(k-1) + W(k)\varepsilon(k)$
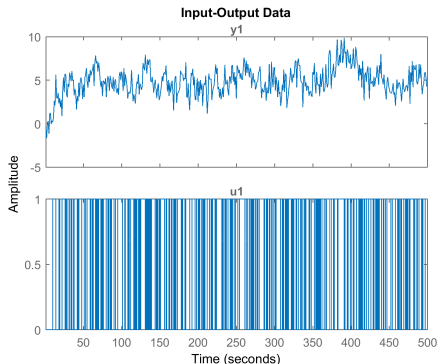**end loop**

Table of contents

Intro & motivation
oooooo

Recursive least-squares and ARX
oooooooooooooooooooo

Recursive instrumental variables
oooooo●oo

OE system

Take the same OE system we used in the ARX example:

$$y(k) = \frac{bq^{-1}}{1 + fq^{-1}} u(k) + e(k), \quad f = -0.9, b = 1$$
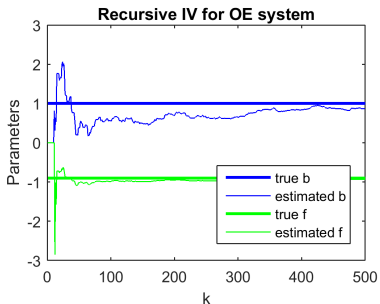
with the same dataset:

Input-Output Data

Results with recursive IV

We use the IV model with colored noise $v(k)$:

$$y(k) + fy(k-1) = bu(k-1) + v(k)$$

Instrument vector: $Z(k) = [u(k-2), u(k-1)]$.
As before, $P^{-1}(0) = \frac{1}{\delta}I$, $\delta = 10^{-3}$.

Recursive IV for OE system

Conclusion: Better than recursive ARX