# System Identification

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# Part IV

## Correlation analysis

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# Motivation 1

## Why other techniques than transient analysis?

Transient analysis of step and impulse responses:

- Only works for a few system orders
- Must usually be done (semi-)manually
- Gives a rough, heuristic model of the system

The upcoming system identification methods:

- Work for arbitrary system orders
- Provide fully implementable, automatic algorithms
- Have solution accuracy guarantees (under appropriate conditions)

# Motivation 2

## Why correlation analysis?

- Closest to transient analysis (model = impulse response)
- True nonparametric model
- “Simple” general identification technique

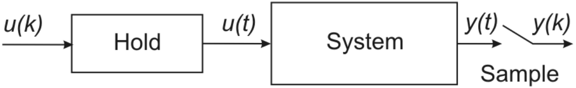
# Classification

Recall **Types of models** from Part I:

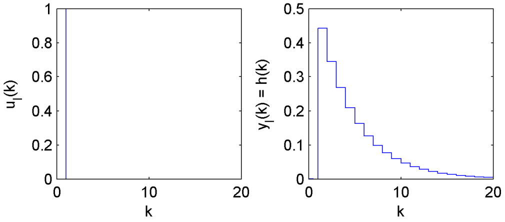
1. Mental or verbal models
2. Graphs and tables (nonparametric)
3. Mathematical models, with two subtypes:
   - First-principles, analytical models
   - Models from system identification

Correlation analysis is truly a nonparametric method; it produces an *impulse response model*.

# Recall: discrete-time model



# Discrete-time impulse response




Discrete-time, unit impulse signal:

$$u_{\mathrm{I}}(k) = \begin{cases} 1 & k = 0 \\ 0 & k > 0 \end{cases}$$

(does not have area 1, so it's different from the discrete-time realization of the continuous-time impulse!)

Discrete-time impulse response:

$$y_I(k) = h(k), \quad k \geq 0$$

$h(k), k \geq 0$ is also called the weighting function of the system.

# Convolution

The (disturbance-free) response to an arbitrary signal $u(k)$ is the *convolution* of the input and the impulse response:

$$y(k) = \sum_{j=0}^{\infty} h(j)u(k-j)$$

Intuition: Consider a signal $\tilde{u}_j(k)$ equal to $u(j)$ at $k = j$, and 0 elsewhere; just a shifted and scaled unit impulse:

$$\tilde{u}_j(k) = u(j)u_{\mathrm{I}}(k-j)$$

So, the response to $\tilde{u}_j(k)$ is a shifted and scaled impulse response:

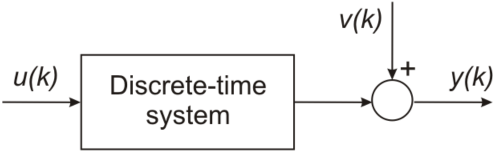
$$\tilde{y}_j(k) = u(j)h(k-j)$$

Now, $u(k)$ is the superposition of all signals $\tilde{u}_j$, and due to linearity:

$$y(k) = \sum_{j=0}^{k} \tilde{y}_j(k) = \sum_{j=0}^{k} u(j)h(k-j) = \sum_{j=0}^{k} h(j)u(k-j) = \sum_{j=0}^{\infty} h(j)u(k-j)$$

where zero initial conditions were assumed, i.e. $u(j) = 0 \forall j < 0$.

# Impulse-response model




$$\boxed{y(k) = \sum_{j=0}^{\infty} h(j)u(k-j) + v(k)}$$

Includes, in addition to the ideal model, a disturbance term $v(k)$.

# Assumptions

## Assumptions

1. The input $u(k)$ is a stationary stochastic process.
2. The input $u(k)$ and the disturbance $v(k)$ are independent.

Recall:

- Independence of random variables.
- Stationary stochastic process: constant mean at every time step, covariance only depends on difference between time steps and not on absolute time.

# Covariance functions

The covariance functions are defined as follows:

$$r_{yu}(\tau) = \mathrm{E}\left\{y(k+\tau)u(k)\right\}$$
$$r_{u}(\tau) = \mathrm{E}\left\{u(k+\tau)u(k)\right\}$$

Note: These quantities are true covariances only when the input and output are zero-mean, so if they are not, then the means must be subtracted prior to applying the algorithm.

# Relationship of covariances and impulse response

If there were no disturbance, then:

$$\begin{aligned} r_{yu}(\tau) &= \mathrm{E}\left\{y(k+\tau)u(k)\right\} \\ &= \mathrm{E}\left\{\left[\sum_{j=0}^{\infty} h(j)u(k+\tau-j)\right]u(k)\right\} \\ &= \sum_{j=0}^{\infty} h(j)\mathrm{E}\left\{u(k+\tau-j)u(k)\right\} = \sum_{j=0}^{\infty} h(j)r_u(\tau-j) \end{aligned}$$

The errors coming from the disturbance are dealt with later, implicitly, using linear regression.

# Impulse response identification

Writing the covariance relationship for all $\tau$:

$$r_{yu}(0) = \sum_{j=0}^{\infty} h(j) r_u(-j) = h(0)r_u(0) + h(1)r_u(-1) + h(2)r_u(-2) + \dots$$

$$r_{yu}(1) = \sum_{j=0}^{\infty} h(j) r_u(1-j) = h(0)r_u(1) + h(1)r_u(0) + h(2)r_u(-1) + \dots$$

$$\dots$$

we obtain (in principle) an infinite system of linear equations:

- Coefficients $r_u(\tau)$, $r_{yu}(\tau)$.
- Unknowns $h(0), h(1), \dots$: solution of the system.

Next, a practical algorithm working with finite data is given.

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# Covariances from data

Consider we are given signals $u(k), y(k)$ with $k = 1, \ldots, N$.
We have, for positive $\tau$:

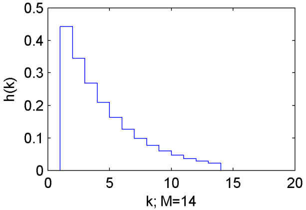
$$\begin{aligned} r_u(\tau) &= \mathrm{E}\left\{u(k+\tau)u(k)\right\} \\ &\approx \boxed{\frac{1}{N}\sum_{k=1}^{N-\tau} u(k+\tau)u(k)} \\ &=: \widehat{r}_u(\tau), \qquad \forall \tau \geq 0 \end{aligned}$$

and $\widehat{r}_u(-\tau) = \widehat{r}_u(\tau)$ for negative values, as $u$ is a stationary process.

For positive and negative $\tau$:

$$\begin{aligned} r_{yu}(\tau) &= \mathrm{E}\left\{y(k+\tau)u(k)\right\} \\ &\approx \boxed{\frac{1}{N}\sum_{k=1-\min\{\tau,0\}}^{N-\max\{\tau,0\}} y(k+\tau)u(k)} \\ &=: \widehat{r}_{yu}(\tau), \qquad \forall \tau \geq 0 \end{aligned}$$

# Finite impulse response model




Impose the condition $h(k) = 0$ for $k \geq M$. We obtain the finite impulse response (FIR) model:

$$y(k) = \sum_{j=0}^{M-1} h(j)u(k-j) + v(k)$$

The covariance relationship is similarly truncated:

$$r_{yu}(\tau) = \sum_{j=0}^{M-1} h(j)r_u(\tau - j)$$

Note: $M$ must be taken so that $MT_s \gg$ dominant time constants (or equivalently, the system is close to steady-state)

# Linear system

Using $\widehat{r}_{yu}$, $\widehat{r}_u$ estimated from data, write the truncated equations for $\tau = 0, \ldots, T-1$ (keeping in mind that $\widehat{r}_u(-\tau) = \widehat{r}_u(\tau)$):

$$
\begin{aligned}
\widehat{r}_{yu}(0) &= \sum_{j=0}^{M-1} h(j)\widehat{r}_u(-j) \\
&= h(0)\widehat{r}_u(0) + h(1)\widehat{r}_u(1) + \ldots + h(M-1)\widehat{r}_u(M-1) \\
\widehat{r}_{yu}(1) &= \sum_{j=0}^{M-1} h(j)\widehat{r}_u(1-j) \\
&= h(0)\widehat{r}_u(1) + h(1)\widehat{r}_u(0) + \ldots + h(M-1)\widehat{r}_u(M-2) \\
\ldots \\
\widehat{r}_{yu}(T-1) &= \sum_{j=0}^{M-1} h(j)\widehat{r}_u(T-1-j) \\
&= h(0)\widehat{r}_u(T-1) + h(1)\widehat{r}_u(T-2) + \ldots + h(M-1)\widehat{r}_u(T-M)
\end{aligned}
$$

– a linear system of $T$ equations in $M$ unknowns $h(0), \ldots, h(M-1)$.

# Linear system (continued)

In matrix form:

$$\boxed{\begin{bmatrix} \widehat{r}_{yu}(0) \\ \widehat{r}_{yu}(1) \\ \vdots \\ \widehat{r}_{yu}(T-1) \end{bmatrix} = \begin{bmatrix} \widehat{r}_u(0) & \widehat{r}_u(1) & \dots & \widehat{r}_u(M-1) \\ \widehat{r}_u(1) & \widehat{r}_u(0) & \dots & \widehat{r}_u(M-2) \\ \vdots & & & \\ \widehat{r}_u(T-1) & \widehat{r}_u(T-2) & \dots & \widehat{r}_u(T-M) \end{bmatrix} \cdot \begin{bmatrix} h(0) \\ h(1) \\ \vdots \\ h(M-1) \end{bmatrix}}$$

Naively taking $T = M$ would give an exact system solution, but due to noise and disturbances this solution would be overfitted. So it is necessary to take $T > M$ (preferably, $T \gg M$).

Then we can apply the machinery of linear regression (see Part 3) to solve this problem.

# Using the FIR model

Once the system has been solved for the estimated $\hat{h}$, we predict outputs with:

$$\boxed{\hat{y}(k) = \sum_{j=0}^{M-1} \hat{h}(j)u(k-j)}$$

# Special case: White noise input

Consider the case when the input $u(k)$ is zero-mean white noise.

Then, $r_u(\tau) = 0$ whenever $\tau \neq 0$ (since white noise is uncorrelated), and $r_{yu}(\tau) = \sum_{j=0}^{\infty} h(j) r_u(\tau - j)$ simplifies to:

$$r_{yu}(\tau) = h(\tau) r_u(0)$$

This leads to the easy algorithm:

$$\hat{h}(\tau) = \frac{\widehat{r}_{yu}(\tau)}{\widehat{r}_u(0)}$$

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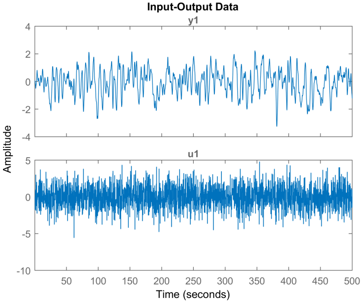
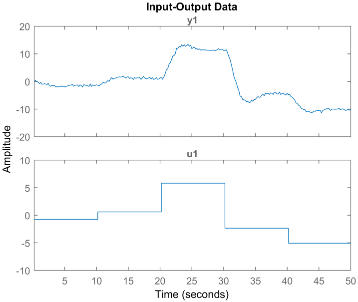
# Experimental data

Consider we are given the following, separate, identification and validation data sets.

`plot(id);` and `plot(val);`

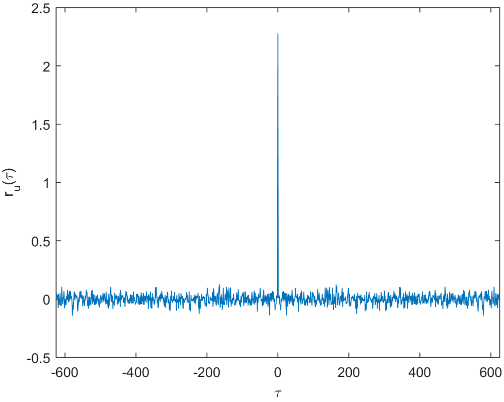



There are 2500 samples in the identification data. We notice that the data is zero-mean.

# Input covariance

`[c, tau] = xcorr(id.u);` and `plot(c, tau);`




The input is white noise.

# Applying correlation analysis

`fir = cra(id, M, 0);` or `fir = cra(id, M, 0, plotlevel);`

Arguments:

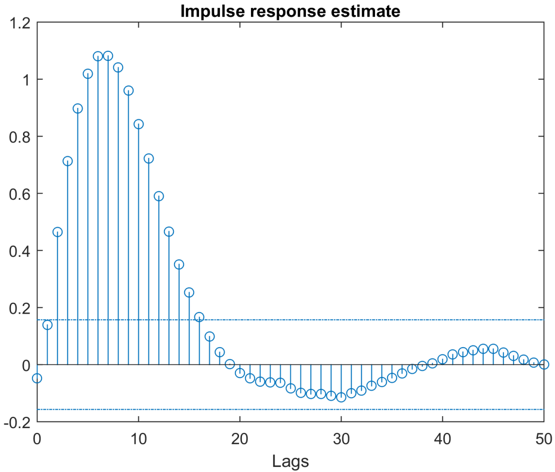
1. Identification data.
2. FIR length $M$, here it is set to 50.
3. Third argument 0 means no *input whitening* is performed.

Dealing with non-ideal inputs:

- If input is not zero-mean, pass the data through `detrend` to remove the means.
- If input is not white noise, the third argument should be left to default (by not specifying it or setting it to an empty matrix), which means input whitening is performed.

# Applying correlation analysis (continued)

By default (or with `plotlevel=1`) the FIR parameters are shown with a 99% confidence interval.

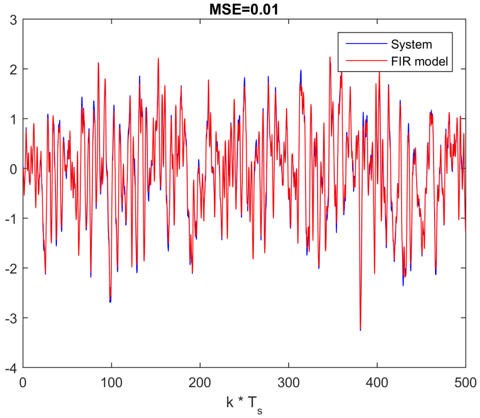



`plotlevel=2` also produces the covariance functions.

# Results on the identification data

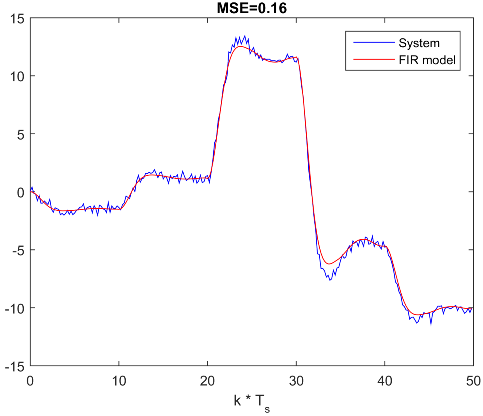
```
yhat = conv(fir, id.u); yhat = yhat(1:length(id.u));
```

To simulate the FIR model, a *convolution* between the FIR parameters and the input is performed. The simulated output is longer than needed so we cut it off at the right length.



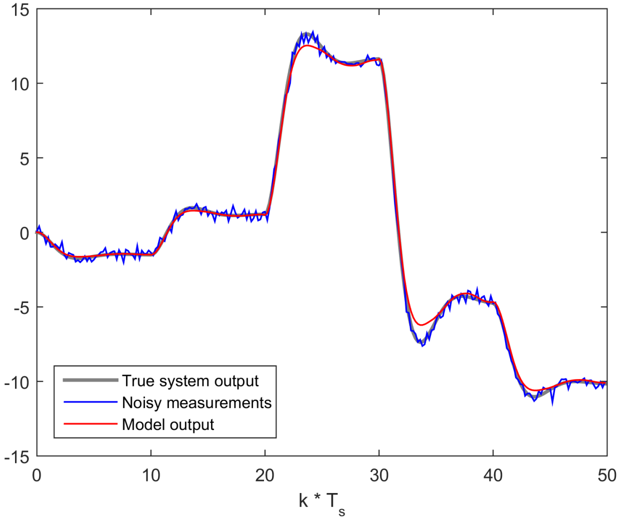
# Validation of the FIR model

```
yhat = conv(fir, val.u); yhat = yhat(1:length(val.u));
```




Results OK, not great.

# Insight into the different signals



# Alternative: `impulseest` function

`model = impulseest(id, M);` or `model = impulseest(id);`

Uses a more involved algorithm than the one studied in the lectures.

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# Simplified guarantee in the white-noise case

### Additional assumption

3. The input $u(k)$ is zero-mean white noise.

### Theorem

In the white-noise case, as the number of data points $N$ grows to infinity, the estimates $\widehat{h}(\tau)$ converge to the true values $h(\tau)$.

Remark: This type of property, where the true solution is obtained in the limit of infinite data, is called *consistency*.