System Identification

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Part X

Model validation and practical issues

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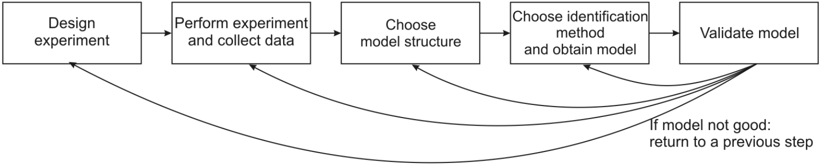
Recall: Importance of validation



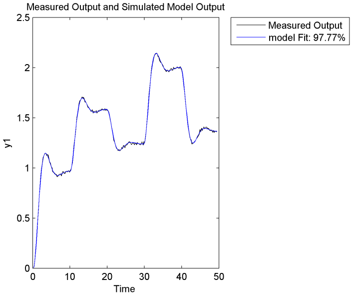
Model validation is a crucial step: the model must be good enough (for its intended usage).

If validation is unsuccessful, previous steps in the workflow must be redone, for instance:

- Rerun the identification algorithm with different parameters (e.g. δ in recursive methods).
- Change the model structure: e.g. orders of polynomials *na*, *nb* in ARX, or even the model type entirely, e.g. IV instead of ARX
- Design and run a new experiment (e.g. more data, different input signal)

Motivation

So far, we validated and selected models mostly informally, by examining plots or comparing errors – using *common sense*.



Next, some mathematically well-founded tests will be given.

However, common sense remains indispensable – mathematical tests work under assumptions that may not always be satisfied.

Focus: Prediction error methods

We focus on *single-output* models obtained by *prediction error methods*.

Some of the tests can be extended to other settings.

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Whiteness: Intuition

Recall general PEM model structure:

$$y(k) = G(q^{-1})u(k) + H(q^{-1})e(k)$$

where $e(k)$ is *assumed* to be zero-mean white noise.

PEM are derived so that the prediction error $\varepsilon(k) = y(k) - \widehat{y}(k) = e(k)$. If the system satisfies the model structure (so the assumption holds), and moreover if the model is accurate, then $\varepsilon(k)$ is also zero-mean white noise.

Whiteness hypothesis

(W) The prediction errors $\varepsilon(k)$ are zero-mean white noise.

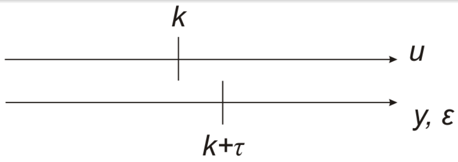
Independence of past inputs: Intuition

$$y(k) = G(q^{-1})u(k) + v(k)$$

If the model G is accurate, it entirely explains the influence of inputs $u(k)$ on current and future outputs $y(k+\tau)$. Therefore, the errors $\varepsilon(k+\tau) = y(k+\tau) - \widehat{y}(k+\tau)$ are only influenced by the disturbances v, and are *independent* of inputs $u(k)$.

Independence hypothesis 1

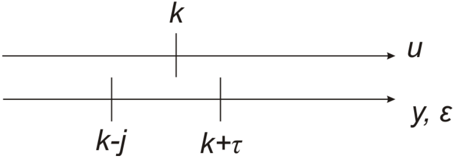
(I1) The prediction errors $\varepsilon(k+\tau)$ are independent of inputs $u(k)$ for $\tau \geq 0$ (i.e., current and future errors are independent of current inputs).



Independence of all inputs: Intuition

$$y(k) = G(q^{-1})u(k) + v(k)$$

If the experiment is closed-loop, $u(k)$ depends on past outputs and this will lead to a correlation of past errors $\varepsilon(k+\tau), \tau < 0$ with $u(k)$ (note the independence for $\tau \geq 0$ is not affected). If open-loop, then $\varepsilon(k+\tau), \tau < 0$ is also independent from $u(k)$.



Independence hypothesis 2

(I2) The prediction errors $\varepsilon(k+\tau)$ are independent of $u(k)$ for *any* τ (i.e., all the errors are independent of all the inputs).

All hypotheses

(W) The prediction errors $\varepsilon(k)$ are zero-mean white noise.

(I1) The prediction errors $\varepsilon(k+\tau)$ are independent of $u(k)$ for $\tau \geq 0$ (current and future errors are independent of current inputs).

(I2) The prediction errors $\varepsilon(k+\tau)$ are independent of $u(k)$ for *any* τ (all the errors are independent of all the inputs).

A good model should satisfy (W) and (I1), and if there is no feedback, also (I2).

We will develop tests that allow to either accept or reject these hypotheses for a given model, and therefore validate or reject the model.

Whiteness: Correlations

Recall the correlation function (equal to the covariance in zero-mean case):

$$r_\varepsilon(\tau) = \mathrm{E}\left\{\varepsilon(k+\tau)\varepsilon(k)\right\}$$

If $\varepsilon(k)$ is zero-mean white noise:

- The correlation function is zero, $r_\varepsilon(\tau) = 0$ for any nonzero τ.
- At zero, $r_\varepsilon(0)$ is the variance σ^2 of the white noise.

Whiteness: Correlations from data

Correlations are estimated from data, and then normalized by the (estimated) variance:

$$\boxed{\begin{aligned} \widehat{r}_\varepsilon(\tau) &= \frac{1}{N} \sum_{k=1}^{N-\tau} \varepsilon(k+\tau)\varepsilon(k) \\ x(\tau) &= \frac{\widehat{r}_\varepsilon(\tau)}{\widehat{r}_\varepsilon(0)} \end{aligned}}$$

Whiteness test

In practice, $x(\tau)$ will never be zero for finite data, so instead we check if it is small for nonzero τ. For statistical reasons, we impose a cutoff at $\frac{1.96}{\sqrt{N}}$

Whiteness test

If $|x(\tau)| \leq \frac{1.96}{\sqrt{N}}$ for all $\tau \neq 0$ supported by the data, then the whiteness hypothesis (W) is accepted. Otherwise, (W) is rejected.

Independence: Correlations

To verify independence of ε from u, use *cross-correlation* function:

$$r_{\varepsilon u}(\tau) = \mathrm{E}\left\{\varepsilon(k+\tau)u(k)\right\}$$

1. If (I1) is true, then $r_{\varepsilon u}(\tau) = 0$ for $\tau \geq 0$.
2. If (I2) is true, then $r_{\varepsilon u}(\tau) = 0$ for any τ.

Estimation from data and normalization:

$$\boxed{\begin{aligned} \widehat{r}_{\varepsilon u}(\tau) &= \begin{cases} \frac{1}{N}\sum_{k=1}^{N-\tau}\varepsilon(k+\tau)u(k) & \text{if } \tau \geq 0 \\ \frac{1}{N}\sum_{k=1-\tau}^{N}\varepsilon(k+\tau)u(k) & \text{if } \tau < 0 \end{cases} \\ x(\tau) &= \frac{\widehat{r}_{\varepsilon u}(\tau)}{\sqrt{\widehat{r}_{\varepsilon}(0)\widehat{r}_{u}(0)}} \end{aligned}}$$

Independence test at τ

Independence tests

If $|x(\tau)| \leq \frac{1.96}{\sqrt{N}}$, $\forall \tau \geq 0$ supported by the data, then the independence hypothesis (I1) is accepted.
If the condition holds $\forall \tau$ supported by the data (including negative τ), then (I2) is also accepted.

If the model is accurate (I1 holds), then checking the condition at $\tau < 0$ (I2) verifies the presence of feedback.

Correlation tests: Overall interpretation

$$y(k) = G(q^{-1})u(k) + H(q^{-1})e(k)$$

- If W and I1 hold, then the entire model (G and H) is correct
- If I1 holds but W fails, then G is correct but H is incorrect
- If I1 holds and I2 fails, there is feedback in the data. If I2 also holds then there is no feedback

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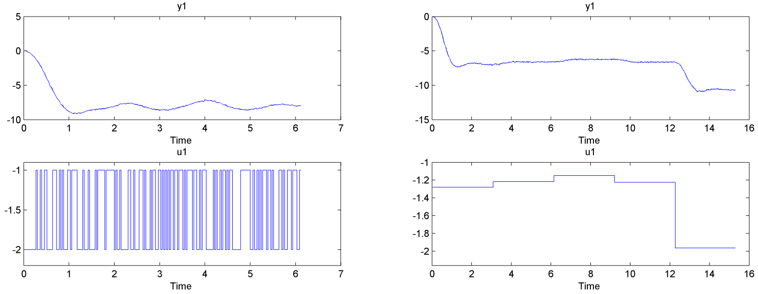
Matlab example: Experimental data

The real system is in output-error form:

$$y(k) = \frac{B(q^{-1})}{F(q^{-1})} u(k) + e(k)$$

and has order $n = 3$.

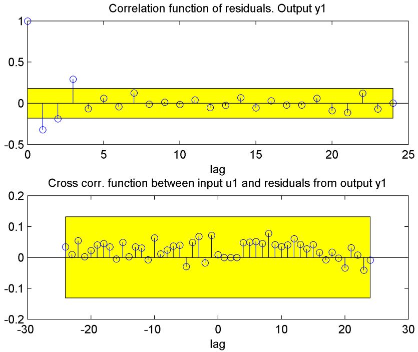
`plot(id);` and `plot(val);`



Matlab: ARX model

First, we try an ARX model:

```
mARX = arx(id, [3, 3, 1]); resid(mARX, id);
```

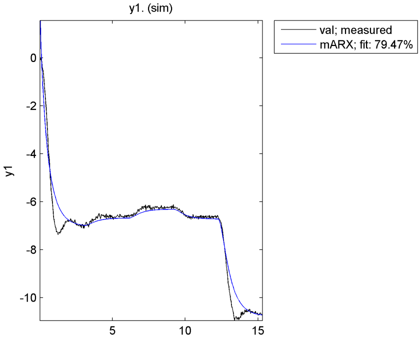


The whiteness test (W) fails, and the model is rejected. This is because the system is not within the model class.

As I1 holds, we conclude that the input-output model G is good, but the noise model H is wrong and we should work to improve that part.

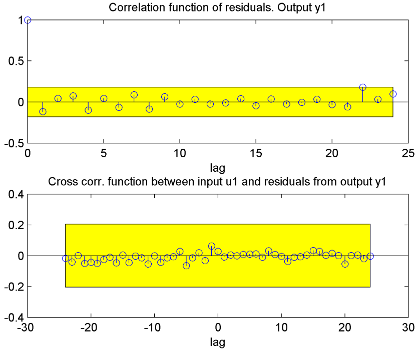
Matlab: ARX model (continued)

Simulating on the validation data confirms the fact that the model is poor.



Matlab: OE model

```
mOE = oe(id, [3, 3, 1]); resid(mOE, id);
```

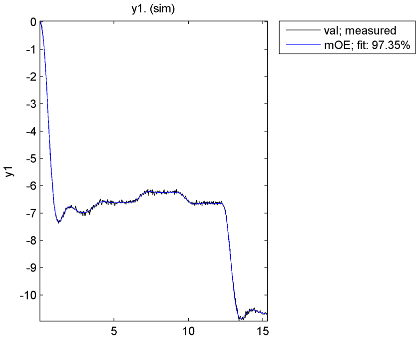


OE model passes all the tests – as expected because the OE model class contains the real system. Thus, both G and H are validated.

Important note: The Matlab functions impose a smaller cutoff for the correlations, so they are less likely to reject a correct model.

Matlab: OE model (continued)

Simulating the model on the validation data confirms the model has good quality.



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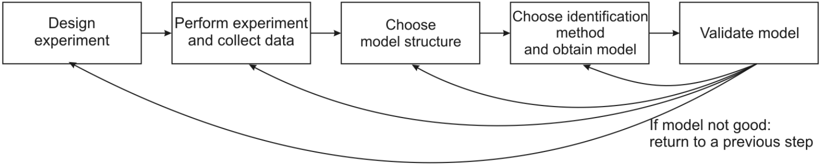
Structure selection in workflow



While we nearly always tuned the model structure (e.g. type, orders, length), the criteria for doing so were often informal.

Next, we discuss structure selection in a formal way.

Structure selection: Model complexity

Consider we are given several model structures $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_\ell$.
Example: ARX structures of increasing order.

How to choose among them?

First idea: choose $\mathcal{M}_i$ leading to the smallest mean squared error:

$$V(\widehat{\theta}) = \frac{1}{N} \sum_{k=1}^{N} \varepsilon(k)^2$$

This ignores the complexity of the model, which is related to:

- the computational effort for identification and simulation
- the amount of data needed for identification
- the risk of overfitting

We explore other options that do consider model complexity (without going into their derivation).

Akaike's information criterion (AIC)

$$W_{\text{AIC}} = N \log V(\widehat{\theta}) + 2p, \text{ or equivalently: } \log V(\widehat{\theta}) + \frac{2p}{N}$$

where N is the number of data points and p the number of parameters (e.g., $na + nb$ in ARX).

Choice: Model with smallest W_{AIC}.

Intuition:

- The term $2p$ penalizes the complexity of the model (number of parameters).
- Division by the number N of data points in $2p/N$ takes into account that more data allows more parameters to be identified.
- Taking the logarithm of the MSE allows to better differentiate between small values of the MSE.

Final prediction error (FPE)

$$W_{\text{FPE}} = V(\widehat{\theta})\frac{1 + p/N}{1 - p/N}$$

Choice: Model with smallest W_{FPE}.

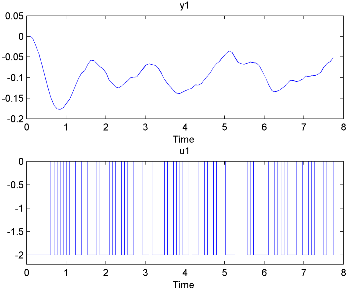
Intuition: When N is large:

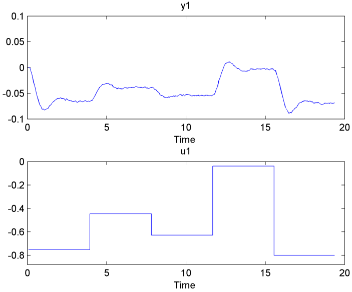
$$V(\widehat{\theta})\frac{1 + p/N}{1 - p/N} = V(\widehat{\theta})(1 + \frac{2p/N}{1 - p/N}) \approx V(\widehat{\theta})(1 + \frac{2p}{N})$$

The term $\frac{2p}{N}$ works like before, but now it leads to a correction proportional to the MSE rather than getting added directly.

Matlab example

An OE system with $n = 2$.





Matlab: `selstruc` with AIC

Recall `arxstruc`:

```
Na = 1:15; Nb = 1:15; Nk = 1:5;
NN = struc(Na, Nb, Nk); V = arxstruc(id, val, NN);
```

- `struc` generates all combinations of orders in `Na, Nb, Nk`.
- `arxstruc` identifies for each combination an ARX model on the data `id`, simulates it on the data `val`, and returns information about the MSEs, model orders etc. in `V`.

Matlab: `selstruc` with AIC (continued)

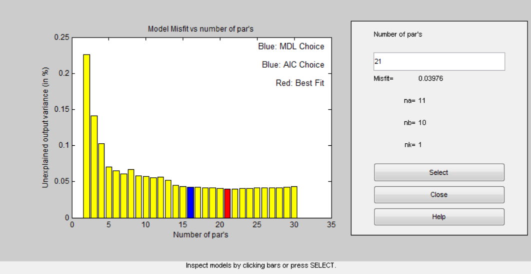
To choose the structure with the Akaike's information criterion:

```
N = selstruc(V, 'aic');
```

For our data, N$= [8, 8, 1]$.

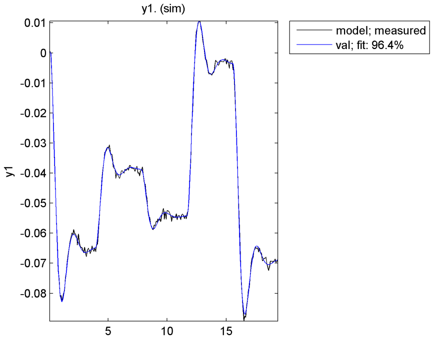
Alternatively, graphical selection also allows using AIC:

```
N = selstruc(V, 'plot');
```



Note that the best-AIC model is not (always) the same as the best-fit model!

Matlab: Results



Remarks

AIC, FPE also work if the system is not in the model class.

Matlab offers functions `aic`, `fpe` that compute these criteria for a list of models with any structure.

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Motivation

Consider a case where *the real system* obeys the ARMAX structure:

$$A_0(q^{-1})y(k) = B_0(q^{-1})u(k) + C_0(q^{-1})e(k)$$

where subscript 0 indicates quantities related to the real system.

This is equivalent to any model:

$$W(q^{-1})A_0(q^{-1})y(k) = W(q^{-1})B_0(q^{-1})u(k) + W(q^{-1})C_0(q^{-1})e(k)$$

with $W(q^{-1})$ some polynomial of order nw.

So, using ARMAX identification with $na = na_0 + nw$, $nb = nb_0 + nw$, $nc = nc_0 + nw$ can produce an accurate model. This model is however too complicated (overparametrized), and will have some nearly common factors $W(q^{-1})$ in all polynomials (only "nearly" because of the approximate nature of the identification).

Pole-zero cancellations

This type of situation can be identified by checking if some poles and zeros of the model (approximately) cancel each other out.

We exemplify using Matlab function `pzmap`, which shows the poles and zeros of G in the generic model:

$$y(k) = G(q^{-1})u(k) + v(k)$$

For the ARMAX example, $G(q^{-1}) = \frac{W(q^{-1})B_0(q^{-1})}{W(q^{-1})A_0(q^{-1})}$, so the roots of W are both poles and zeros and (approximately) cancel each other out.

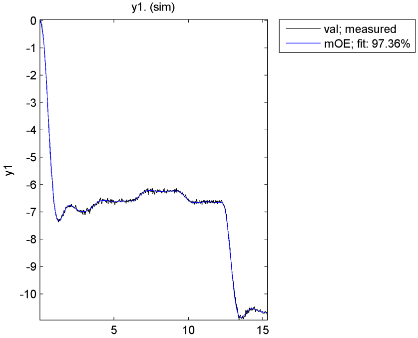
This idea extends to other model types besides ARMAX.

Matlab: overparameterized OE model

On the same data as for correlation tests (recall system has order $n = 3$):

```
mOE = oe(id, [5, 5, 1]);
```

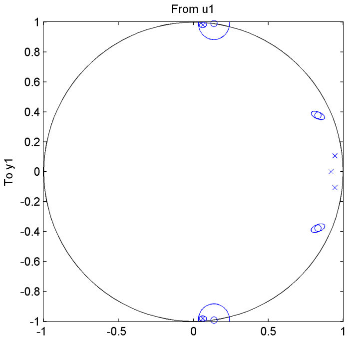
Looking at the validation data, the model is accurate:



Matlab: testing for pole-zero cancellations

```
pzmap(mOE, 'sd', nsd);
```

Arguments `'sd', nsd` specify a statistical confidence region around the poles and zeros. Here we take `nsd`=1.96.



Two pairs of poles and zeros have overlapping confidence regions ⇒ likely they are canceling each other. This indicates the order should be 3 (the true system order).

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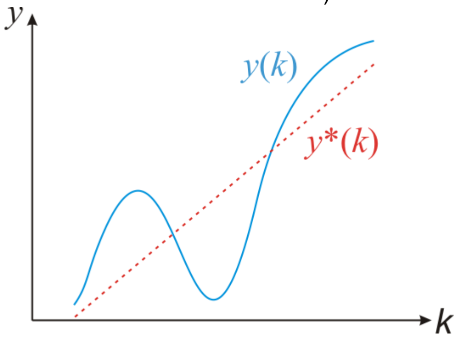
Drifts

Sometimes, the data will contain spurious slow signals called *drifts*, coming e.g. from slow disturbances (as opposed to the fast noise or disturbance, which we know how to handle)



Idea: Treat the drifts as time series, fit them with linear regression, and remove them

Estimating drifts

1. Treat the input and output as separate time series, write drift models:

$$u^*(k) = \theta_1^u + \theta_2^u k + \theta_3^u k^2 + \ldots + \theta_n^u k^{n-1}$$
$$y^*(k) = \theta_1^y + \theta_2^y k + \theta_3^y k^2 + \ldots + \theta_n^y k^{n-1}$$

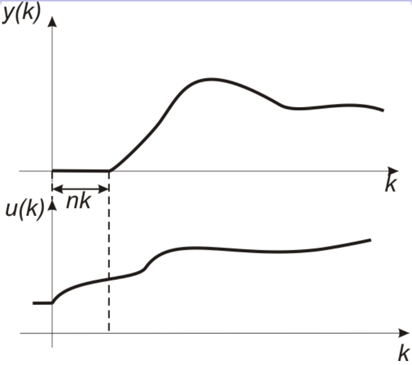
2. Find the parameter vectors θ^u, θ^y by linear regression on $u(k), y(k)$ and compute the corresponding drifts $u^*(k), y^*(k)$
3. Subtract the drifts from the data:

$$\bar{u}(k) = u(k) - u^*(k), \quad \bar{y}(k) = y(k) - y^*(k)$$

4. Identify as usual, but with “detrended” signals $\bar{u}, \bar{y}$

Notes: Matlab function `detrend` available; Removing zero-order drifts = removing the means

Time delays

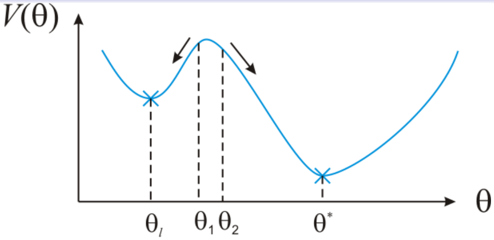


Read the delay on the graph. Set *nk* appropriately in Matlab, or otherwise add *nk* leading zeros to the polynomial $B(q^{-1})$ in the model:

$$\cdots y(k) = \frac{B(q^{-1})}{\cdots} u(k) + \cdots$$

Note: Taking *nk* too small is safe (possibly requiring an increase of *nb*); too large and it breaks the model!

Local minima

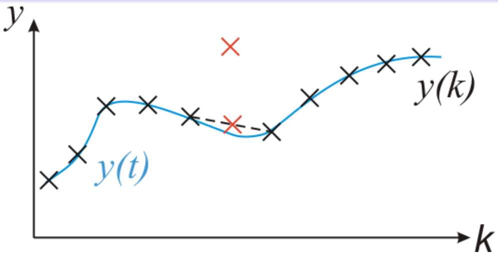


- Iterative optimization (needed for methods that cannot be solved as linear regression, such as ARMAX and OE) may get stuck in local minima
- E.g. if initialized to θ_1, Newton's method likely converges to the local minimum θ_ℓ. But from θ_2 it finds the global optimum θ^*!

$\Rightarrow$ If result is bad and local minima suspected, restart the optimization from another initial parameter vector

Note: ARMAX usually converges to the global optimum; OE often to local optima except when u is white noise

Outliers



- Sometimes, a few measurements will be wildly incorrect, due to e.g. transient malfunctions. These are called outliers
- Best tested via the prediction error ε, after finding an initial model: if $\varepsilon(k)$ is anomalously large at some step k, an outlier is likely
- Solution 1: Fill in the data using e.g. the average of $y(k-1)$ and $y(k+1)$ (shown in the figure), or the model prediction $\hat{y}(k)$
- Solution 2: Cap the prediction error at a reasonable maximum $\varepsilon_{\max}$, so $V(\theta) = \sum_{k=1}^{N} \min\{\varepsilon^2(k), \varepsilon_{\max}\}$