

System Identification

Control Engineering EN, 3rd year B.Sc.
Technical University of Cluj-Napoca
Romania

Lecturer: Lucian Buşoniu



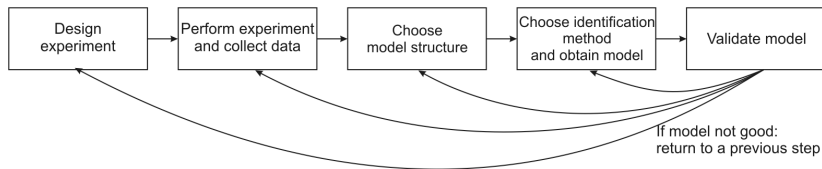
Part X

Model validation and practical issues

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Recall: Importance of validation



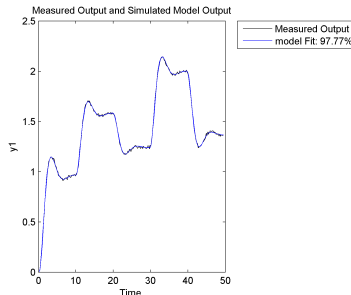
Model validation is a crucial step: the model must be good enough (for its intended usage).

If validation is unsuccessful, previous steps in the workflow must be redone, for instance:

- Rerun the identification algorithm with different parameters (e.g. δ in recursive methods).
- Change the model structure: e.g. orders of polynomials n_a, n_b in ARX, or even the model type entirely, e.g. IV instead of ARX
- Design and run a new experiment (e.g. more data, different input signal)

Motivation

So far, we validated and selected models mostly informally, by examining plots or comparing errors – using *common sense*.



Next, some mathematically well-founded tests will be given.

However, common sense remains indispensable – mathematical tests work under assumptions that may not always be satisfied.

Focus: Prediction error methods

We focus on *single-output* models obtained by *prediction error methods*.

Some of the tests can be extended to other settings.

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Whiteness: Intuition

Recall general PEM model structure:

$$y(k) = G(q^{-1})u(k) + H(q^{-1})e(k)$$

where $e(k)$ is *assumed* to be zero-mean white noise.

PEM are derived so that the prediction error $\varepsilon(k) = y(k) - \hat{y}(k) = e(k)$. If the system satisfies the model structure (so the assumption holds), and moreover if the model is accurate, then $\varepsilon(k)$ is also zero-mean white noise.

Whiteness hypothesis

(W) The prediction errors $\varepsilon(k)$ are zero-mean white noise.

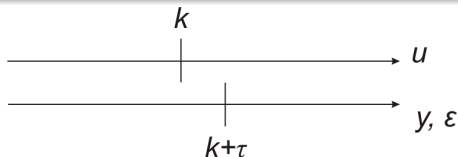
Independence of past inputs: Intuition

$$y(k) = G(q^{-1})u(k) + v(k)$$

If the model G is accurate, it entirely explains the influence of inputs $u(k)$ on current and future outputs $y(k + \tau)$. Therefore, the errors $\varepsilon(k + \tau) = y(k + \tau) - \hat{y}(k + \tau)$ are only influenced by the disturbances v , and are *independent* of inputs $u(k)$.

Independence hypothesis 1

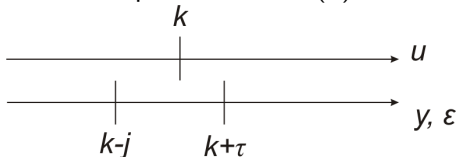
(I1) The prediction errors $\varepsilon(k + \tau)$ are independent of inputs $u(k)$ for $\tau \geq 0$ (i.e., current and future errors are independent of current inputs).



Independence of all inputs: Intuition

$$y(k) = G(q^{-1})u(k) + v(k)$$

If the experiment is closed-loop, $u(k)$ depends on past outputs and this will lead to a correlation of past errors $\varepsilon(k + \tau)$, $\tau < 0$ with $u(k)$ (note the independence for $\tau \geq 0$ is not affected). If open-loop, then $\varepsilon(k + \tau)$, $\tau < 0$ is also independent from $u(k)$.



Independence hypothesis 2

(I2) The prediction errors $\varepsilon(k + \tau)$ are independent of $u(k)$ for any τ (i.e., all the errors are independent of all the inputs).

All hypotheses

- (W) The prediction errors $\varepsilon(k)$ are zero-mean white noise.
- (I1) The prediction errors $\varepsilon(k + \tau)$ are independent of $u(k)$ for $\tau \geq 0$ (current and future errors are independent of current inputs).
- (I2) The prediction errors $\varepsilon(k + \tau)$ are independent of $u(k)$ for *any* τ (all the errors are independent of all the inputs).

A good model should satisfy (W) and (I1), and if there is no feedback, also (I2).

We will develop tests that allow to either accept or reject these hypotheses for a given model, and therefore validate or reject the model.

Whiteness: Correlations

Recall the correlation function (equal to the covariance in zero-mean case):

$$r_{\varepsilon}(\tau) = \mathbb{E} \{ \varepsilon(k + \tau) \varepsilon(k) \}$$

If $\varepsilon(k)$ is zero-mean white noise:

- The correlation function is zero, $r_{\varepsilon}(\tau) = 0$ for any nonzero τ .
- At zero, $r_{\varepsilon}(0)$ is the variance σ^2 of the white noise.

Whiteness: Correlations from data

Correlations are estimated from data, and then normalized by the (estimated) variance:

$$\hat{r}_\varepsilon(\tau) = \frac{1}{N} \sum_{k=1}^{N-\tau} \varepsilon(k+\tau)\varepsilon(k)$$
$$x(\tau) = \frac{\hat{r}_\varepsilon(\tau)}{\hat{r}_\varepsilon(0)}$$

Whiteness test

In practice, $x(\tau)$ will never be zero for finite data, so instead we check if it is small for nonzero τ . For statistical reasons, we impose a cutoff at $\frac{1.96}{\sqrt{N}}$

Whiteness test

If $|x(\tau)| \leq \frac{1.96}{\sqrt{N}}$ for all $\tau \neq 0$ supported by the data, then the whiteness hypothesis (W) is accepted. Otherwise, (W) is rejected.

Independence: Correlations

To verify independence of ε from u , use *cross-correlation* function:

$$r_{\varepsilon u}(\tau) = \mathbb{E} \{ \varepsilon(k + \tau) u(k) \}$$

- 1 If (I1) is true, then $r_{\varepsilon u}(\tau) = 0$ for $\tau \geq 0$.
- 2 If (I2) is true, then $r_{\varepsilon u}(\tau) = 0$ for any τ .

Estimation from data and normalization:

$$\hat{r}_{\varepsilon u}(\tau) = \begin{cases} \frac{1}{N} \sum_{k=1}^{N-\tau} \varepsilon(k + \tau) u(k) & \text{if } \tau \geq 0 \\ \frac{1}{N} \sum_{k=1-\tau}^N \varepsilon(k + \tau) u(k) & \text{if } \tau < 0 \end{cases}$$

$$x(\tau) = \frac{\hat{r}_{\varepsilon u}(\tau)}{\sqrt{\hat{r}_{\varepsilon}(\tau) \hat{r}_u(\tau)}}$$

Independence test at τ

Independence tests

If $|x(\tau)| \leq \frac{1.96}{\sqrt{N}}$, $\forall \tau \geq 0$ supported by the data, then the independence hypothesis (I1) is accepted.

If the condition holds $\forall \tau$ supported by the data (including negative τ), then (I2) is also accepted.

If the model is accurate (I1 holds), then checking the condition at $\tau < 0$ (I2) verifies the presence of feedback.

Correlation tests: Overall interpretation

$$y(k) = G(q^{-1})u(k) + H(q^{-1})e(k)$$

- If W and $I1$ hold, then the entire model (G and H) is correct
- If $I1$ holds but W fails, then G is correct but H is incorrect
- If $I1$ holds and $I2$ fails, there is feedback in the data. If $I2$ also holds then there is no feedback

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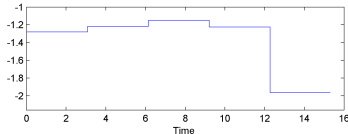
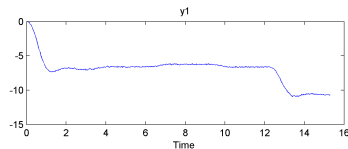
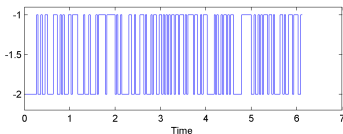
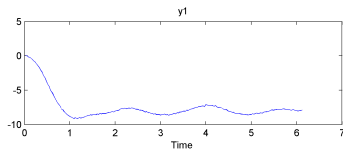
Matlab example: Experimental data

The real system is in output-error form:

$$y(k) = \frac{B(q^{-1})}{F(q^{-1})}u(k) + e(k)$$

and has order $n = 3$.

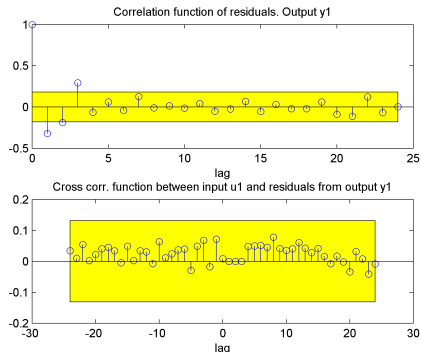
`plot(id);` and `plot(val);`



Matlab: ARX model

First, we try an ARX model:

```
mARX = arx(id, [3, 3, 1]); resid(mARX, id);
```

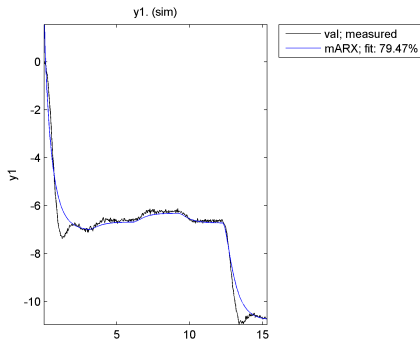


The whiteness test (W) fails, and the model is rejected. This is because the system is not within the model class.

As I1 holds, we conclude that the input-output model G is good, but the noise model H is wrong and we should work to improve that part.

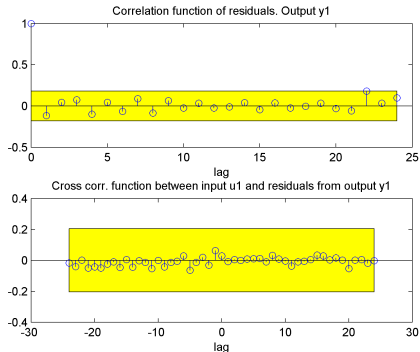
Matlab: ARX model (continued)

Simulating on the validation data confirms the fact that the model is poor.



Matlab: OE model

```
mOE = oe(id, [3, 3, 1]); resid(mOE, id);
```



OE model passes all the tests – as expected because the OE model class contains the real system. Thus, both G and H are validated.

Important note: The Matlab functions impose a smaller cutoff for the correlations, so they are less likely to reject a correct model.

Matlab: OE model (continued)

Simulating the model on the validation data confirms the model has good quality.

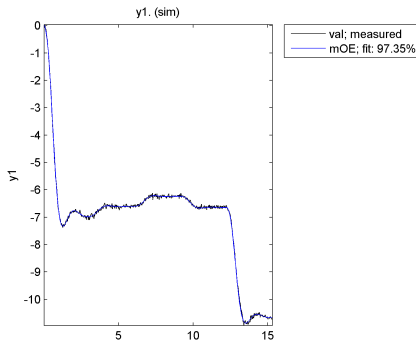
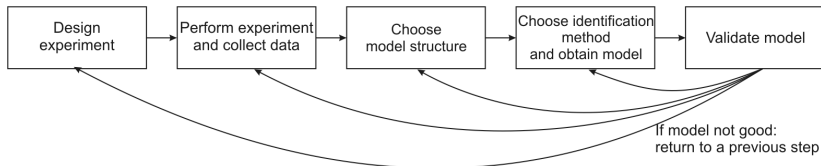


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Structure selection in workflow



While we nearly always tuned the model structure (e.g. type, orders, length), the criteria for doing so were often informal.

Next, we discuss structure selection in a formal way.

Structure selection: Model complexity

Consider we are given several model structures $\mathcal{M}_1, \mathcal{M}_2, \dots, \mathcal{M}_\ell$.

Example: ARX structures of increasing order.

How to choose among them?

First idea: choose \mathcal{M}_i leading to the smallest mean squared error:

$$V(\hat{\theta}) = \frac{1}{N} \sum_{k=1}^N \varepsilon(k)^2$$

This ignores the complexity of the model, which is related to:

- the computational effort for identification and simulation
- the amount of data needed for identification
- the risk of overfitting

We explore other options that do consider model complexity (without going into their derivation).

Akaike's information criterion (AIC)

$$W_{\text{AIC}} = N \log V(\hat{\theta}) + 2p, \text{ or equivalently: } \log V(\hat{\theta}) + \frac{2p}{N}$$

where N is the number of data points and p the number of parameters (e.g., $na + nb$ in ARX).

Choice: Model with smallest W_{AIC} .

Intuition:

- The term $2p$ penalizes the complexity of the model (number of parameters).
- Division by the number N of data points in $2p/N$ takes into account that more data allows more parameters to be identified.
- Taking the logarithm of the MSE allows to better differentiate between small values of the MSE.

Final prediction error (FPE)

$$W_{\text{FPE}} = V(\hat{\theta}) \frac{1 + p/N}{1 - p/N}$$

Choice: Model with smallest W_{FPE} .

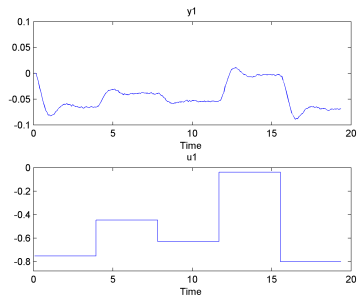
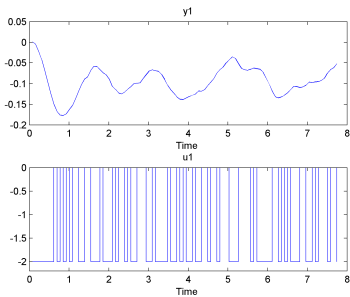
Intuition: When N is large:

$$V(\hat{\theta}) \frac{1 + p/N}{1 - p/N} = V(\hat{\theta}) \left(1 + \frac{2p/N}{1 - p/N}\right) \approx V(\hat{\theta}) \left(1 + \frac{2p}{N}\right)$$

The term $\frac{2p}{N}$ works like before, but now it leads to a correction proportional to the MSE rather than getting added directly.

Matlab example

An OE system with $n = 2$.



Matlab: selstruc with AIC

Recall `arxstruc`:

```
Na = 1:15; Nb = 1:15; Nk = 1:5;  
NN = struc(Na, Nb, Nk); V = arxstruc(id, val, NN);
```

- `struc` generates all combinations of orders in `Na`, `Nb`, `Nk`.
- `arxstruc` identifies for each combination an ARX model on the data `id`, simulates it on the data `val`, and returns information about the MSEs, model orders etc. in `V`.

Matlab: `selstruc` with AIC (continued)

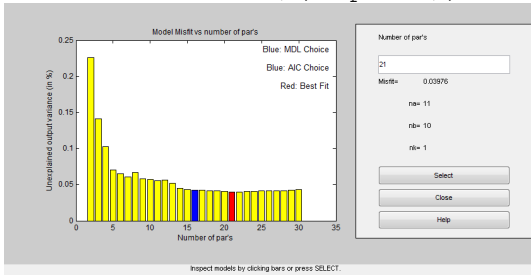
To choose the structure with the Akaike's information criterion:

```
N = selstruc(V, 'aic');
```

For our data, $N = [8, 8, 1]$.

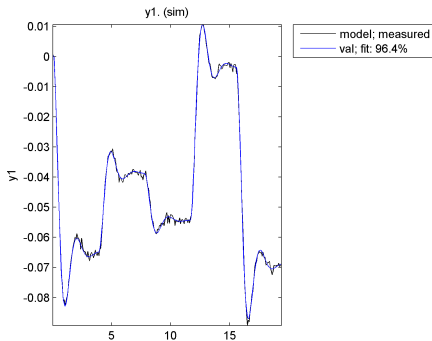
Alternatively, graphical selection also allows using AIC:

```
N = selstruc(V, 'plot');
```



Note that the best-AIC model is not (always) the same as the best-fit model!

Matlab: Results



Remarks

AIC, FPE also work if the system is not in the model class.

Matlab offers functions `aic`, `fpe` that compute these criteria for a list of models with any structure.

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Motivation

Consider a case where *the real system* obeys the ARMAX structure:

$$A_0(q^{-1})y(k) = B_0(q^{-1})u(k) + C_0(q^{-1})e(k)$$

where subscript 0 indicates quantities related to the real system.

This is equivalent to any model:

$$W(q^{-1})A_0(q^{-1})y(k) = W(q^{-1})B_0(q^{-1})u(k) + W(q^{-1})C_0(q^{-1})e(k)$$

with $W(q^{-1})$ some polynomial of order nw .

So, using ARMAX identification with $na = na_0 + nw$, $nb = nb_0 + nw$, $nc = nc_0 + nw$ can produce an accurate model. This model is however **too complicated** (overparametrized), and will have some nearly common factors $W(q^{-1})$ in all polynomials (only “nearly” because of the approximate nature of the identification).

Pole-zero cancellations

This type of situation can be identified by checking if some poles and zeros of the model (approximately) cancel each other out.

We exemplify using Matlab function `pzmap`, which shows the poles and zeros of G in the generic model:

$$y(k) = G(q^{-1})u(k) + v(k)$$

For the ARMAX example, $G(q^{-1}) = \frac{W(q^{-1})B_0(q^{-1})}{W(q^{-1})A_0(q^{-1})}$, so the roots of W are both poles and zeros and (approximately) cancel each other out.

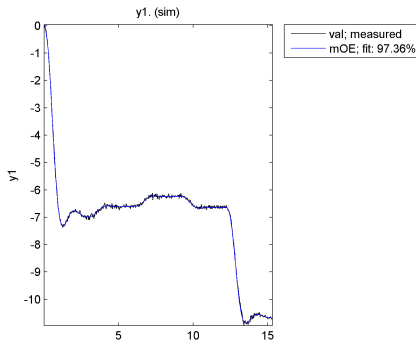
This idea extends to other model types besides ARMAX.

Matlab: overparameterized OE model

On the same data as for correlation tests (recall system has order $n = 3$):

```
mOE = oe(id, [5, 5, 1]);
```

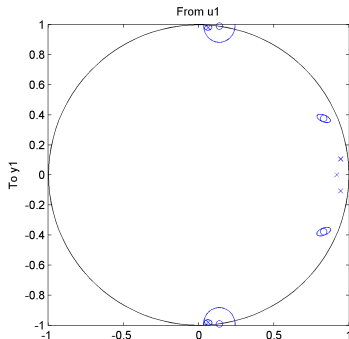
Looking at the validation data, the model is accurate:



Matlab: testing for pole-zero cancellations

```
pzmap(mOE, 'sd', nsd);
```

Arguments 'sd', nsd specify a statistical confidence region around the poles and zeros. Here we take $nsd=1.96$.



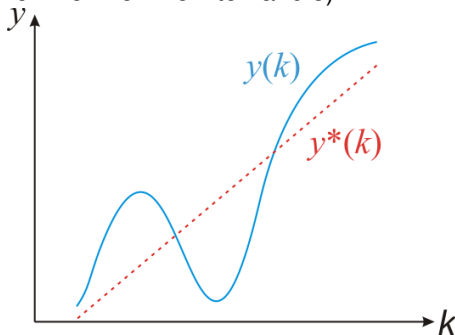
Two pairs of poles and zeros have overlapping confidence regions \Rightarrow likely they are canceling each other. This indicates the order should be 3 (the true system order).

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 - Time delays
 - Local minima
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Drifts

Sometimes, the data will contain spurious slow signals called *drifts*, coming e.g. from slow disturbances (as opposed to the fast noise or disturbance, which we know how to handle)



Idea: Treat the drifts as time series, fit them with linear regression, and remove them

Estimating drifts

- 1 Treat the input and output as separate time series, write drift models:

$$u^*(k) = \theta_1^u + \theta_2^u k + \theta_3^u k^2 + \dots + \theta_n^u k^{n-1}$$

$$y^*(k) = \theta_1^y + \theta_2^y k + \theta_3^y k^2 + \dots + \theta_n^y k^{n-1}$$

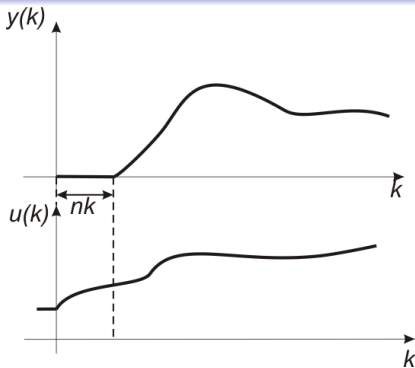
- 2 Find the parameter vectors θ^u, θ^y by linear regression on $u(k), y(k)$ and compute the corresponding drifts $u^*(k), y^*(k)$
- 3 Subtract the drifts from the data:

$$\bar{u}(k) = u(k) - u^*(k), \quad \bar{y}(k) = y(k) - y^*(k)$$

- 4 Identify as usual, but with “detrended” signals \bar{u}, \bar{y}

Note: Removing zero-order drifts = removing the means

Time delays

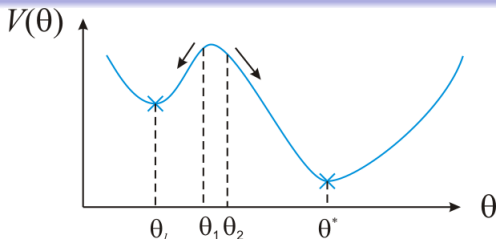


Read the delay on the graph. Set nk appropriately in Matlab, or otherwise add nk leading zeros to the polynomial $B(q^{-1})$ in the model:

$$\dots y(k) = \frac{B(q^{-1})}{\dots} u(k) + \dots$$

Note: Taking nk too small is safe; too large and it breaks the model!

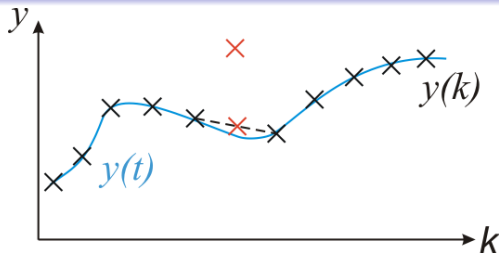
Local minima



- Iterative optimization (needed for methods that cannot be solved as linear regression, such as ARMAX and OE) may get stuck in local minima
 - E.g. if initialized to θ_1 , Newton's method likely converges to the local minimum θ_ℓ . But from θ_2 it finds the global optimum θ^* !
- ⇒ If result is bad and local minima suspected, restart the optimization from another initial parameter vector

Note: ARMAX usually converges to the global optimum; OE often to local optima except when u is white noise

Outliers



- Sometimes, a few measurements will be wildly incorrect, due to e.g. transient malfunctions. These are called outliers
- Best tested via the prediction error ε , after finding an initial model: if $\varepsilon(k)$ is anomalously large at some step k , an outlier is likely
- **Solution 1:** Fill in the data using e.g. the average of $y(k-1)$ and $y(k+1)$ (shown in the figure), or the model prediction $\hat{y}(k)$
- **Solution 2:** Cap the prediction error at a reasonable maximum ε_{\max} , so $V(\theta) = \sum_{k=1}^N \min\{\varepsilon^2(k), \varepsilon_{\max}\}$