

# System Identification

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## Part VIII

# Instrumental variable methods. Closed-loop identification

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- 1 Analytical development of instrumental variable methods
- 2 Matlab example
- 3 Theoretical guarantees
- 4 Closed-loop identification using IV

# Classification

- 1 Mental or verbal models
- 2 Graphs and tables (nonparametric)
- 3 Mathematical models, with two subtypes:
  - First-principles, analytical models
  - **Models from system identification**

Like prediction error methods, instrumental variable methods produce *parametric*, polynomial models.

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# Overall motivation

- The ARX method is simple (linear regression), but only supports limited classes of disturbance
- General PEM supports any (reasonable) disturbance, but it is numerically difficult to apply

Can we come up with a method that combines both advantages?

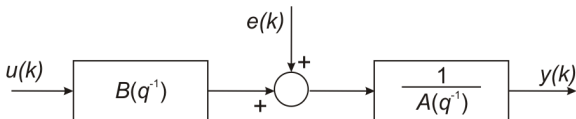
**Yes! Instrumental variables**

# Recall ARX model

$$A(q^{-1})y(k) = B(q^{-1})u(k) + e(k)$$

$$(1 + a_1q^{-1} + \dots + a_{na}q^{-na})y(k) =$$

$$(b_1q^{-1} + \dots + b_{nb}q^{-nb})u(k) + e(k)$$



In explicit form:

$$y(k) + a_1y(k-1) + a_2y(k-2) + \dots + a_{na}y(k-na)$$

$$= b_1u(k-1) + b_2u(k-2) + \dots + b_{nb}u(k-nb) + e(k)$$

where the model parameters are:  $a_1, a_2, \dots, a_{na}$  and  $b_1, b_2, \dots, b_{nb}$ .

# Linear regression representation

$$\begin{aligned}
 y(k) &= -a_1 y(k-1) - a_2 y(k-2) - \dots - a_{na} y(k-na) \\
 &\quad b_1 u(k-1) + b_2 u(k-2) + \dots + b_{nb} u(k-nb) + e(k) \\
 &= [-y(k-1) \quad \dots \quad -y(k-na) \quad u(k-1) \quad \dots \quad u(k-nb)] \\
 &\quad \cdot [a_1 \quad \dots \quad a_{na} \quad b_1 \quad \dots \quad b_{nb}]^\top + e(k) \\
 &=: \varphi^\top(k) \theta + e(k)
 \end{aligned}$$

**Regressor vector:**  $\varphi \in \mathbb{R}^{na+nb}$ , previous output and input values.

**Parameter vector:**  $\theta \in \mathbb{R}^{na+nb}$ , polynomial coefficients.



# Identification problem and solution

Given dataset  $u(k), y(k), k = 1, \dots, N$ , find model parameters  $\theta$  to achieve small equation errors  $\varepsilon(k)$  in:

$$y(k) = \varphi^\top(k)\theta + \varepsilon(k)$$

**Formal objective:** minimize the mean squared error:

$$V(\theta) = \frac{1}{N} \sum_{k=1}^N \varepsilon(k)^2$$

**Solution:** can be written in several ways, here we use:

$$\hat{\theta} = \left[ \frac{1}{N} \sum_{k=1}^N \varphi(k)\varphi^\top(k) \right]^{-1} \left[ \frac{1}{N} \sum_{k=1}^N \varphi(k)y(k) \right]$$

# Theoretical guarantee

Recall that for the guarantees, a true parameter vector  $\theta_0$  is assumed to exist:

$$y(k) = \varphi^\top(k)\theta_0 + v(k)$$

Analyze the parameter errors (a vector of  $n$  elements):

$$\begin{aligned} \hat{\theta} - \theta_0 &= \left[ \frac{1}{N} \sum_{k=1}^N \varphi(k)\varphi^\top(k) \right]^{-1} \left[ \frac{1}{N} \sum_{k=1}^N \varphi(k)y(k) \right] \\ &- \left[ \frac{1}{N} \sum_{k=1}^N \varphi(k)\varphi^\top(k) \right]^{-1} \left[ \frac{1}{N} \sum_{k=1}^N \varphi(k)\varphi^\top(k) \right] \theta_0 \\ &= \left[ \frac{1}{N} \sum_{k=1}^N \varphi(k)\varphi^\top(k) \right]^{-1} \left[ \frac{1}{N} \sum_{k=1}^N \varphi(k)[y(k) - \varphi^\top(k)\theta_0] \right] \\ &= \left[ \frac{1}{N} \sum_{k=1}^N \varphi(k)\varphi^\top(k) \right]^{-1} \left[ \frac{1}{N} \sum_{k=1}^N \varphi(k)v(k) \right] \end{aligned}$$

## Further assumptions

We wish the algorithm to be consistent: the parameter errors should become 0 in the limit of infinite data (and they should be well-defined).

As  $N \rightarrow \infty$ :

$$\frac{1}{N} \sum_{k=1}^N \varphi(k) \varphi^{\top}(k) \rightarrow \mathbb{E} \{ \varphi(k) \varphi^{\top}(k) \}$$

$$\frac{1}{N} \sum_{k=1}^N \varphi(k) v(k) \rightarrow \mathbb{E} \{ \varphi(k) v(k) \}$$

For the error to be (1) well-defined and (2) equal to zero, we need:

- 1  $\mathbb{E} \{ \varphi(k) \varphi^{\top}(k) \}$  invertible.
- 2  $\mathbb{E} \{ \varphi(k) v(k) \}$  zero.

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# Motivation: ARX requires white noise

- We have  $E \{ \varphi(k)v(k) \} = 0$  if the elements of  $\varphi(k)$  are uncorrelated with  $v(k)$  (note that  $v(k)$  is assumed zero-mean).
- But  $\varphi(k)$  includes  $y(k-1), y(k-2), \dots$ , which depend on  $v(k-1), v(k-2), \dots$ !
- So the only option is to have  $v(k)$  uncorrelated with  $v(k-1), v(k-2), \dots \Rightarrow v(k)$  must be *white noise*.

**Instrumental variables** are a solution to remove this limitation to white noise.

# Intuition

$$\hat{\theta} - \theta_0 = \left[ \frac{1}{N} \sum_{k=1}^N \varphi(k) \varphi^T(k) \right]^{-1} \left[ \frac{1}{N} \sum_{k=1}^N \varphi(k) v(k) \right]$$

**Idea:** What if a different vector than  $\varphi(k)$  could be included in the parameter errors?

$$\hat{\theta} - \theta_0 = \left[ \frac{1}{N} \sum_{k=1}^N Z(k) \varphi^T(k) \right]^{-1} \left[ \frac{1}{N} \sum_{k=1}^N Z(k) v(k) \right]$$

where the elements of  $Z(k)$  are uncorrelated with  $v(k)$ . Then  $E\{Z(k)v(k)\} = 0$  and the error can be zero.

Vector  $Z(k)$  has  $n$  elements, which are called **instruments**.

# Instrumental variable method

In order to have:

$$\hat{\theta} - \theta_0 = \left[ \frac{1}{N} \sum_{k=1}^N Z(k) \varphi^\top(k) \right]^{-1} \left[ \frac{1}{N} \sum_{k=1}^N Z(k) v(k) \right] \quad (8.1)$$

the estimated parameter must be:

$$\hat{\theta} = \left[ \frac{1}{N} \sum_{k=1}^N Z(k) \varphi^\top(k) \right]^{-1} \left[ \frac{1}{N} \sum_{k=1}^N Z(k) y(k) \right] \quad (8.2)$$

This  $\hat{\theta}$  is the solution to the system of  $n$  equations:

$$\left[ \frac{1}{N} \sum_{k=1}^N Z(k) [\varphi^\top(k) \theta - y(k)] \right] = 0 \quad (8.3)$$

Constructing and solving this system gives the **basic instrumental variable (IV) method**.

**Exercise:** Show that (8.3) implies (8.2), and that (8.2) implies (8.1).

# Simple instruments

So far the instruments  $Z(k)$  were not discussed. They are usually created based on the inputs (including outputs would lead to correlation with  $v$  and so eliminate the advantage of IV).

**Simple possibility:** just include additional delayed inputs to obtain a vector of the appropriate size,  $n = na + nb$ :

$$Z(k) = [u(k - nb - 1), \dots, u(k - na - nb), u(k - 1), \dots, u(k - nb)]^T$$

Compare to original vector:

$$\varphi(k) = [-y(k - 1), \dots, -y(k - na), u(k - 1), \dots, u(k - nb)]^T$$



# Generalization

Pass the input through a transfer function:

$$\begin{aligned}
 C(q^{-1})x(k) &= D(q^{-1})u(k) \\
 (1 + c_1q^{-1} + \dots + c_{nb}q^{-nc})x(k) &= \\
 (d_1q^{-1} + \dots + d_{nd}q^{-nd})u(k) & \\
 x(k) &= -c_1x(k-1) - c_2x(k-2) - \dots - c_{nc}x(k-nc) \\
 &\quad d_1u(k-1) + d_2u(k-2) + \dots + d_{nd}u(k-nd)
 \end{aligned}$$

and take  $na$  past values from the output  $x$ :

$$Z(k) = [-x(k-1), \dots, -x(k-na), u(k-1), \dots, u(k-nb)]^T$$

**Remark:**  $C(q^{-1})$ ,  $D(q^{-1})$  have different meanings than in PEM.

# Generalized instruments: obtaining the simple case

In order to obtain:

$$Z(k) = [u(k - nb - 1), \dots, u(k - na - nb), u(k - 1), \dots, u(k - nb)]^T$$

set  $C = 1$ ,  $D = -q^{-nb}$ .

**Exercise:** Verify that the desired  $Z(k)$  is indeed obtained.

# Generalized instruments: Initial model

Generalized instruments:

$$Z(k) = [-x(k-1), \dots, -x(k-na), u(k-1), u(k-2), \dots, u(k-nb)]^T$$

Compare to original vector:

$$\varphi(k) = [-y(k-1), \dots, -y(k-na), u(k-1), \dots, u(k-nb)]^T$$

**Idea:** Take instrument generator equal to an initial model,  
 $C(q^{-1}) = \hat{A}(q^{-1})$ ,  $D(q^{-1}) = \hat{B}(q^{-1})$ . This model can be obtained e.g.  
 with ARX estimation.

The instruments are an approximation of  $y$ :

$$Z(k) = [-\hat{y}(k-1), \dots, -\hat{y}(k-na), u(k-1), \dots, u(k-nb)]^T$$

that is *uncorrelated* with the noise.

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# Comparison

Both PEM and IV can be seen as extensions of ARX:

$$A(q^{-1})y(k) = B(q^{-1})u(k) + e(k)$$

to disturbances  $v(k)$  different from white noise  $e(k)$ .

- **PEM** explicitly include the disturbance model in the structure, e.g. in ARMAX  $v(k) = C(q^{-1})e(k)$  leading to  $A(q^{-1})y(k) = B(q^{-1})u(k) + C(q^{-1})e(k)$ .
- **IV methods** do *not* explicitly model the disturbance, but are designed to be resilient to non-white, “colored” disturbance, by using instruments  $Z(k)$  uncorrelated with it.

## Comparison (continued)

**Advantage of IV:** Simple model structure, identification consists only of solving a system of linear equations. In contrast, PEM required solving optimization problems with Newton's method, was susceptible to local minima etc.

**Disadvantage of IV:** In practice, for finite number  $N$  of data, model quality depends heavily on the choice of instruments  $Z(k)$ . Moreover, the resulting model has a larger risk of being unstable (even with a stable real system).

Methods exist to choose instruments  $Z(k)$  that are optimal in a certain sense, but they will not be discussed here.

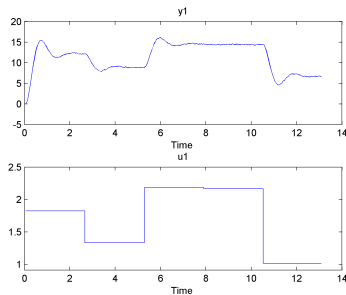
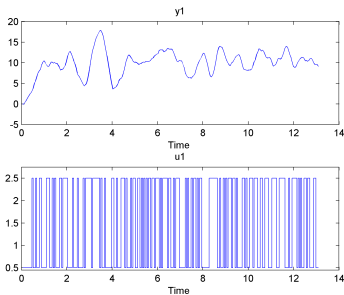
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# Experimental data

Separate identification and validation data sets:

```
plot(id); and plot(val);
```



From prior knowledge, the system has order 2 and the disturbance is colored (does not obey the ARX model structure).

**Remarks:** As before, the identification input is a pseudo-random binary signal, and the validation input a sequence of steps.



# IV identification with custom instruments

Define the instruments by the generating transfer function, using polynomials  $C(q^{-1})$  and  $D(q^{-1})$ .

```
model = iv(id, [na, nb, nk], C, D);
```

Arguments:

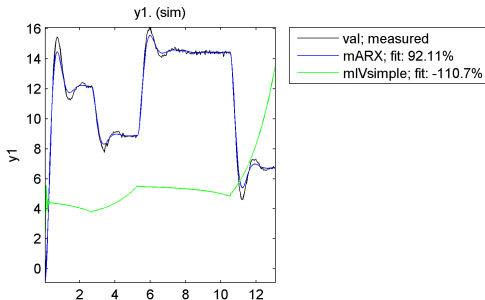
- 1 Identification data.
- 2 Array containing the orders of  $A$  and  $B$  and the delay  $nk$  (like for ARX).
- 3 Polynomials  $C$  and  $D$ , as vectors of coefficients in increasing power of  $q^{-1}$ .

# Result with simple instruments

Take  $C(q^{-1}) = 1$ ,  $D(q^{-1}) = -q^{-nb}$ , leading to

$$Z(k) = [u(k - nb - 1), \dots, u(k - na - nb), u(k - 1), \dots, u(k - nb)]^T.$$

Compare to ARX.



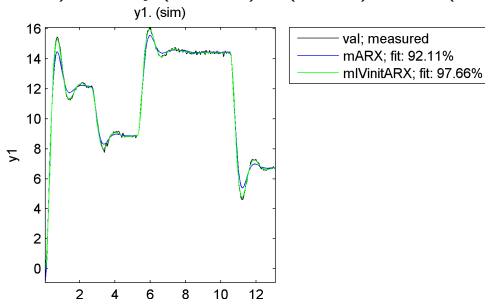
## Conclusions:

- Model unstable  $\Rightarrow$  in general, must pay attention because IV models **are not guaranteed to be stable!** (recall the Comparison)
- Results very bad with this simple choice.

# Result with ARX-model instruments

Take  $C(q^{-1}) = \hat{A}(q^{-1})$ ,  $D(q^{-1}) = \hat{B}(q^{-1})$  from the ARX experiment, leading to

$$Z(k) = [-\hat{y}(k-1), \dots, -\hat{y}(k-na), u(k-1), \dots, u(k-nb)]^T.$$

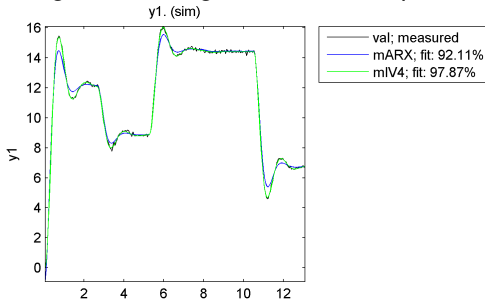


**Conclusion:** IV obtains better results. This is because the disturbance is colored, and IV can deal effectively with this case (whereas ARX cannot – but it still provides a useful starting point for IV).

# Result with automatic instruments

```
model = iv4(id, [na, nb, nk]);
```

Implements an algorithm that generates near-optimal instruments.



**Conclusion:** Virtually the same performance as ARX instruments.

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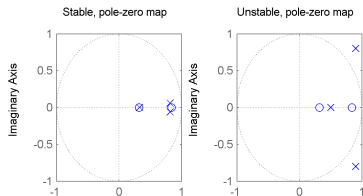
# Assumptions

## Assumptions (simplified)

- 1 The disturbance  $v(k) = H(q^{-1})e(k)$  where  $e(k)$  is zero-mean white noise,  $H(q^{-1})$  is a transfer function satisfying certain conditions.
- 2 The input signal  $u(k)$  has a sufficiently large order of PE and does not depend on the disturbance (the experiment is open-loop).
- 3 The real system is stable and *uniquely* representable by the model chosen: there exists exactly one  $\theta_0$  so that polynomials  $A(q^{-1}; \theta_0)$  and  $B(q^{-1}; \theta_0)$  are identical to those of the real system.
- 4 Matrix  $E \{Z(k)Z^T(k)\}$  is invertible.

# Discussion of assumptions

- Assumption 1 shows the main advantage of IV over PEM: the disturbance can be colored.
- Assumptions 2 and 3 are not very different from those made by PEM. Stability of a discrete-time system requires its poles to be strictly inside the unit circle:



**Question:** Why is the experiment not allowed to be closed-loop?

- Assumption 4 is required to solve the linear system, and given an input with sufficient order of PE boils down to an appropriate selection of instruments (e.g. not repeating the same delayed input  $u(k - i)$  twice).

# Guarantee

## Theorem 1

As the number of data points  $N \rightarrow \infty$ , the solution  $\hat{\theta}$  of IV estimation converges to the true parameter vector  $\theta_0$ .

**Remark:** This is a **consistency** guarantee, in the limit of infinitely many data points.



# Possible extensions

- Multiple-input, multiple-output systems.
- Larger-dimension instruments  $Z$  than parameter vectors  $\theta$  — with other modifications, called extended IV methods.
- Identification of systems operating in closed loop: next

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# Motivation

In practice, systems must often be controlled, because when they operate on their own, in open loop:

- They would be unstable
- Safety or economical limits for the signals would not be satisfied

This means that  $u(k)$  is computed using feedback from  $y(k)$ : the system operates in closed loop

# Closed-loop identification

However, most of the techniques that we studied assume the system functions in open loop! For instance, IV guarantees require (among other things):

- ...
- The input signal  $u(k)$  does not depend on the disturbance (the experiment is open-loop)
- ...

Removing this condition leads to **closed-loop identification**.

Several techniques can be modified for this setting, notably including prediction error methods.

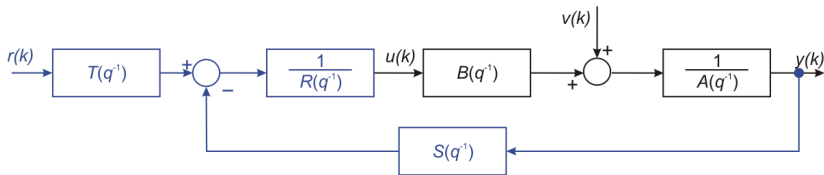
Here, we will focus on IV methods since they are easy to modify.

# Closed-loop IV structure

$$A(q^{-1})y(k) = B(q^{-1})u(k) + v(k)$$

$$R(q^{-1})u(k) = T(q^{-1})r(k) - S(q^{-1})y(k)$$

$R, T, S$  polynomials



Therefore, in general  $u(k)$  depends:

- dynamically (via  $R(q^{-1})$ ),
- on the system output with negative feedback through  $S(q^{-1})$ ,
- and through  $T(q^{-1})$  on an external input  $r(k)$   
– usually a reference signal

# Challenge

The open-loop condition will of course fail. Let us dig deeper into it.

The underlying reason for which we needed the loop open was to make the parameter errors:

$$\hat{\theta} - \theta_0 = \left[ \frac{1}{N} \sum_{k=1}^N Z(k) \varphi^T(k) \right]^{-1} \left[ \frac{1}{N} \sum_{k=1}^N Z(k) v(k) \right]$$

equal to zero, leading to a good model. For this, we require:

- $E \{ Z(k) v(k) \}$  zero.
- $E \{ Z(k) \varphi^T(k) \}$  invertible.

With the usual IV choices, computed based on  $u$  (which now depends on  $y$  and hence on  $v$ ), the first condition would fail.

# Closed-loop IV idea

The vector of IVs  $Z(k)$  is not allowed to depend on  $u(k)$  anymore.

Idea: **make it a function of  $r(k)$** !

Then:

- $E\{Z(k)v(k)\}$  will naturally be zero, since we are the ones generating the reference  $r$ , independently from the disturbance  $v$
- We can make  $E\{Z(k)\varphi^\top(k)\}$  invertible by ensuring the IVs are good (e.g. no linear dependence), and that the reference  $r$  has a sufficiently high order of PE

## Example choices of IVs

Simplest idea – include in  $Z$  the appropriate number of delayed reference values:

$$Z(k) = [r(k-1), r(k-2), \dots, r(k-na-nb)]^T$$

Slightly generalized to linear combinations of these values:

$$Z(k) = F \cdot [r(k-1), r(k-2), \dots, r(k-na-nb)]^T$$

where  $F$  is invertible. The simple case is recovered by taking  $F$  the identity matrix.