System Identification

Control Engineering EN, 3rd year B.Sc.
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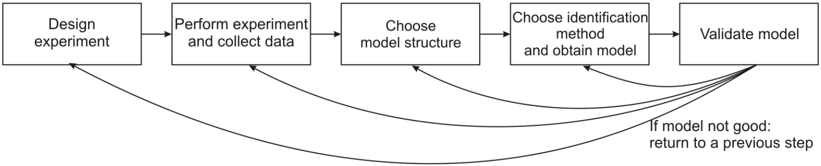
Part IX

Model validation and structure selection

Recall: Importance of validation



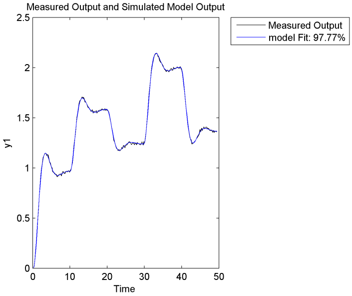
Model validation is a crucial step: the model must be good enough (for its intended usage).

If validation is unsuccessful, previous steps in the workflow must be redone, for instance:

- Rerun the identification algorithm with different parameters (e.g. δ in recursive methods).
- Change the model structure: e.g. orders of polynomials *na*, *nb* in ARX, or even the model type entirely, e.g. IV instead of ARX
- Design and run a new experiment (e.g. more data, different input signal)

Motivation

So far, we validated and selected models mostly informally, by examining plots or comparing errors – using *common sense*.



Next, some mathematically well-founded tests will be given.

However, common sense remains indispensable – mathematical tests work under assumptions that may not always be satisfied.

Focus: Prediction error methods

This lecture focuses on *single-output* models obtained by *prediction error methods*.

Some of the tests can be extended to other settings.

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Whiteness: Intuition

Recall general PEM model structure:

$$y(k) = G(q^{-1})u(k) + H(q^{-1})e(k)$$

where $e(k)$ is *assumed* to be zero-mean white noise.

PEM are derived so that the prediction error $\varepsilon(k) = y(k) - \widehat{y}(k) = e(k)$. If the system satisfies the model structure (so the assumption holds), and moreover if the model is accurate, then $\varepsilon(k)$ is also zero-mean white noise.

Whiteness hypothesis

(W) The prediction errors $\varepsilon(k)$ are zero-mean white noise.

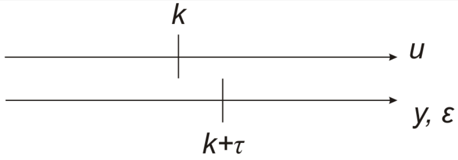
Independence of past inputs: Intuition

$$y(k) = G(q^{-1})u(k) + v(k)$$

If the model G is accurate, it entirely explains the influence of inputs $u(k)$ on current and future outputs $y(k+\tau)$. Therefore, the errors $\varepsilon(k+\tau) = y(k+\tau) - \widehat{y}(k+\tau)$ are only influenced by the disturbances v, and are *independent* of inputs $u(k)$.

Independence hypothesis 1

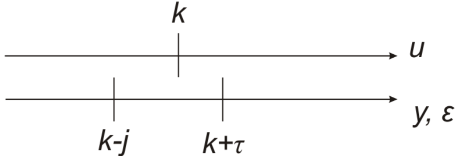
(I1) The prediction errors $\varepsilon(k+\tau)$ are independent of inputs $u(k)$ for $\tau \geq 0$ (i.e., current and future errors are independent of current inputs).



Independence of all inputs: Intuition

$$y(k) = G(q^{-1})u(k) + v(k)$$

If the experiment is closed-loop, $u(k)$ depends on past outputs and this will lead to a correlation of past errors $\varepsilon(k+\tau), \tau < 0$ with $u(k)$ (note the independence for $\tau \geq 0$ is not affected). If open-loop, then $\varepsilon(k+\tau), \tau < 0$ is also independent from $u(k)$.



Independence hypothesis 2

(I2) The prediction errors $\varepsilon(k+\tau)$ are independent of $u(k)$ for *any* τ (i.e., all the errors are independent of all the inputs).

All hypotheses

- (W) The prediction errors $\varepsilon(k)$ are zero-mean white noise.
- (I1) The prediction errors $\varepsilon(k+\tau)$ are independent of $u(k)$ for $\tau \geq 0$ (current and future errors are independent of current inputs).
- (I2) The prediction errors $\varepsilon(k+\tau)$ are independent of $u(k)$ for *any* τ (all the errors are independent of all the inputs).

A good model should satisfy (W) and (I1), and if there is no feedback, also (I2).

We will develop tests that allow to either accept or reject these hypotheses for a given model, and therefore validate or reject the model.

Whiteness: Correlations

Recall the correlation function (equal to the covariance in zero-mean case):

$$r_\varepsilon(\tau) = \mathrm{E}\left\{\varepsilon(k+\tau)\varepsilon(k)\right\}$$

If $\varepsilon(k)$ is zero-mean white noise:

- The correlation function is zero, $r_\varepsilon(\tau) = 0$ for any nonzero τ.
- At zero, $r_\varepsilon(0)$ is the variance σ^2 of the white noise.

Whiteness: Correlations from data

Estimating correlations from data:

$$\widehat{r}_\varepsilon(\tau) = \frac{1}{N} \sum_{k=1}^{N-\tau} \varepsilon(k+\tau)\varepsilon(k)$$

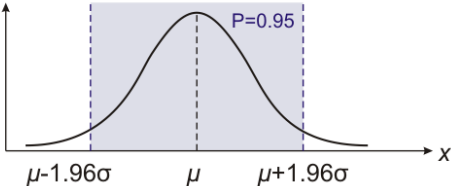
Normalization by the (estimated) variance:

$$x(\tau) = \frac{\widehat{r}_\varepsilon(\tau)}{\widehat{r}_\varepsilon(0)}$$

Then, $x(\tau)$ should be small for nonzero τ (it will usually not be exactly zero for finite data).

Whiteness test

For any Gaussian distribution $\mathcal{N}(\mu, \sigma^2)$, the probability of drawing a sample in the interval $\mu - 1.96\sigma, \mu + 1.96\sigma$ is found (from the Gaussian formula) to be 0.95.



It can be shown that for large N, $x(\tau)$ is distributed according to Gaussian $\mathcal{N}(0, \frac{1}{N})$, and therefore that:

$$P(|x(\tau)| \leq \frac{1.96}{\sqrt{N}}) = 0.95$$

Whiteness test (continued)

Whiteness test

If $|x(\tau)| \leq \frac{1.96}{\sqrt{N}}$ for all $\tau \neq 0$ supported by the data, then the whiteness hypothesis (W) is accepted. Otherwise, (W) is rejected.

Intuition: If the test succeeds, then the hypothesis is likely to be true.

More detailed intuition

Define the following logical propositions:

P1: The model is correct.

P2: The prediction errors are white noise.

P3: The correlation $x(\tau)$ is Gaussian with $\mathcal{N}(0, \frac{1}{N})$.

P4: $|x(\tau)| \leq \frac{1.96}{\sqrt{N}}, \quad \forall \tau \neq 0.$

We have $P1 \Rightarrow P2 \Rightarrow P3$, so when $P3$ is false, $\neg P3 \Rightarrow \neg P2 \Rightarrow \neg P1$ and the model is rejected.

The implication from $P3$ to $P4$ is probabilistic so things become more complicated, but informally speaking we decide that $\neg P3$ when $\neg P4$, i.e., when $|x(\tau)| > \frac{1.96}{\sqrt{N}}$ for at least one $\tau \neq 0$. If $P3$ is true, there is still a small, but acceptable probability of $\neg P4$ occurring, and therefore of rejecting the hypothesis when it was in fact true.

Independence: Correlations

To verify independence of ε from u, use *cross-correlation* function:

$$r_{\varepsilon u}(\tau) = \mathrm{E}\left\{\varepsilon(k+\tau)u(k)\right\}$$

1. If (I1) is true, then $r_{\varepsilon u}(\tau) = 0$ for $\tau \geq 0$.
2. If (I2) is true, then $r_{\varepsilon u}(\tau) = 0$ for any τ.

Estimation from data and normalization:

$$\widehat{r}_{\varepsilon u}(\tau) = \begin{cases} \frac{1}{N}\sum_{k=1}^{N-\tau} \varepsilon(k+\tau)u(k) & \text{if } \tau \geq 0 \\ \frac{1}{N}\sum_{k=1-\tau}^{N} \varepsilon(k+\tau)u(k) & \text{if } \tau < 0 \end{cases}$$

$$x(\tau) = \frac{\widehat{r}_{\varepsilon u}(\tau)}{\sqrt{\widehat{r}_{\varepsilon}(0)\widehat{r}_{u}(0)}}$$

Independence test at τ

Like before, for large N, $x(\tau)$ is distributed according to Gaussian $\mathcal{N}(0, \frac{1}{N})$, and therefore:

$$P(|x(\tau)| \leq \frac{1.96}{\sqrt{N}}) = 0.95$$

Independence tests

If $|x(\tau)| \leq \frac{1.96}{\sqrt{N}}$, $\forall \tau \geq 0$ supported by the data, then the independence hypothesis (I1) is accepted.
If the condition holds $\forall \tau$ supported by the data (including negative τ), then (I2) is also accepted.

In practice, if the model is accurate (I1 holds), then checking the condition at $\tau < 0$ (I2) verifies the presence of feedback.

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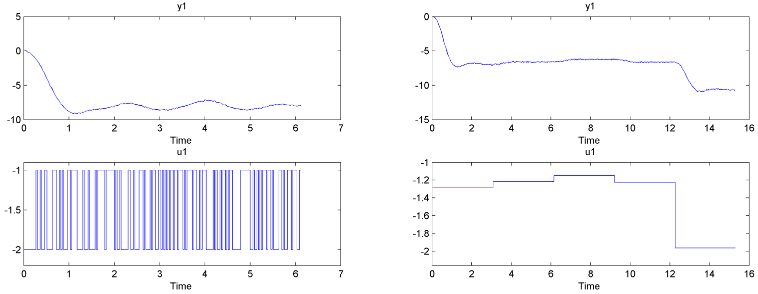
Matlab example: Experimental data

The real system is in output-error form:

$$y(k) = \frac{B(q^{-1})}{F(q^{-1})}u(k) + e(k)$$

and has order $n = 3$.

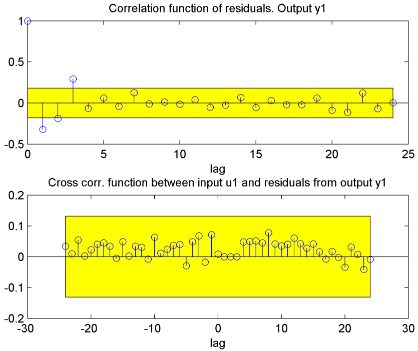
`plot(id);` and `plot(val);`



Matlab: ARX model

First, we try an ARX model:

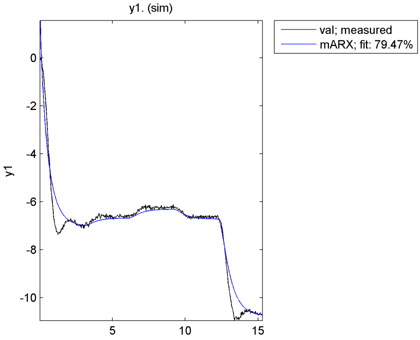
```
mARX = arx(id, [3, 3, 1]); resid(mARX, id);
```



The model fails the whiteness test (W). This is because the system is not within the model class, leading to an inaccurate model. The model is rejected.

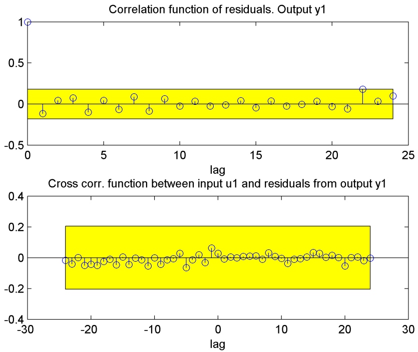
Matlab: ARX model (continued)

Simulating on the validation data confirms the fact that the model is poor.



Matlab: OE model

```
mOE = oe(id, [3, 3, 1]); resid(mOE, id);
```

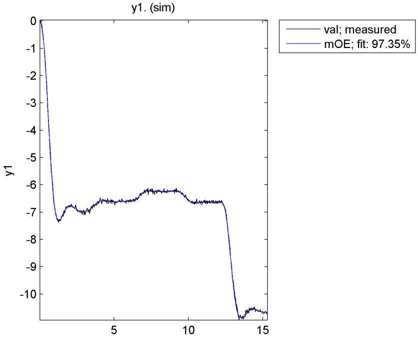


The OE model passes all the tests – as expected because the OE model class contains the real system. The model is therefore validated.

Important note: The Matlab functions use 0.99 probability instead of 0.95, so they are even less likely of rejecting a correct model.

Matlab: OE model (continued)

Simulating the model on the validation data confirms the model has good quality.



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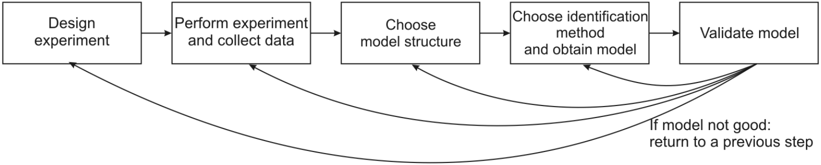
Structure selection in workflow



While we nearly always tuned the model structure (e.g. type, orders, length), the criteria for doing so were often informal.

Next, we discuss structure selection in a formal way.

Consider we are given several model structures $\mathcal{M}_1, \mathcal{M}_2, \dots, \mathcal{M}_\ell$.
Example: ARX structures of increasing order.

How to choose among them?

First idea: choose $\mathcal{M}_i$ leading to the smallest mean squared error:

$$V(\widehat{\theta}) = \frac{1}{N} \sum_{k=1}^{N} \varepsilon(k)^2$$

This does not take into account the complexity of the model, so it ignores the computational effort needed to simulate it, and also has a risk of overfitting. We explore other options that do consider model complexity (without going into their mathematical derivation).

Akaike's information criterion (AIC)

$$W_{\text{AIC}} = N \log V(\widehat{\theta}) + 2p, \text{ or equivalently: } \log V(\widehat{\theta}) + \frac{2p}{N}$$

where N is the number of data points and p the number of parameters (e.g., $na + nb$ in ARX).

Choice: Model with smallest W_{AIC}.

Intuition:

- The term $2p$ penalizes the complexity of the model (number of parameters).
- Division by the number N of data points in $2p/N$ takes into account that more data allows more parameters to be identified.
- Taking the logarithm of the MSE allows to better differentiate between small values of the MSE.

Final prediction error (FPE)

$$W_{\text{FPE}} = V(\widehat{\theta})\frac{1 + p/N}{1 - p/N}$$

Choice: Model with smallest W_{FPE}.

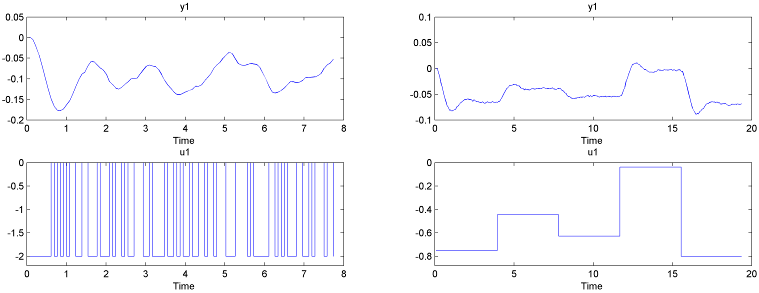
Intuition: When N is large:

$$V(\widehat{\theta})\frac{1 + p/N}{1 - p/N} = V(\widehat{\theta})(1 + \frac{2p/N}{1 - p/N}) \approx V(\widehat{\theta})(1 + \frac{2p}{N})$$

The term $\frac{2p}{N}$ works like before, but now it *multiplies* the MSE rather than getting added.

Matlab example

An OE system with $n = 2$.



Matlab: `selstruc` with AIC

Recall `arxstruc`:

```
Na = 1:15; Nb = 1:15; Nk = 1:5;
NN = struc(Na, Nb, Nk); V = arxstruc(id, val, NN);
```

- `struc` generates all combinations of orders in `Na, Nb, Nk`.
- `arxstruc` identifies for each combination an ARX model on the data `id`, simulates it on the data `val`, and returns information about the MSEs, model orders etc. in `V`.

Matlab: `selstruc` with AIC (continued)

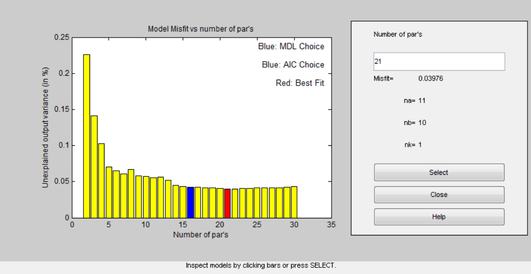
To choose the structure with the Akaike's information criterion:

```
N = selstruc(V, 'aic');
```

For our data, `N`$= [8, 8, 1]$.

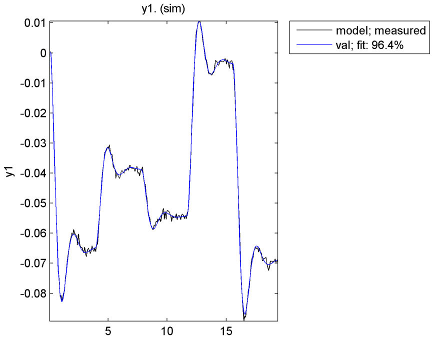
Alternatively, graphical selection also allows using AIC:

```
N = selstruc(V, 'plot');
```



Note that the best-AIC model is not (always) the same as the best-fit model!

Matlab: Results



Remarks

AIC, FPE also work if the system is not in the model class.

Matlab offers functions `aic, fpe` that compute these criteria for a list of models.

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Motivation

Consider *the real system* obeys the ARMAX structure:

$$A_0(q^{-1})y(k) = B_0(q^{-1})u(k) + C_0(q^{-1})e(k)$$

where subscript 0 indicates quantities related to the real system.

This is equivalent to any model:

$$W(q^{-1})A_0(q^{-1})y(k) = W(q^{-1})B_0(q^{-1})u(k) + W(q^{-1})C_0(q^{-1})e(k)$$

with $W(q^{-1})$ some polynomial of order nw.

So, using ARMAX identification with $na = na_0 + nw$, $nb = nb_0 + nw$, $nc = nc_0 + nw$ can produce an accurate model. This model is however too complicated (overparametrized), and will have some nearly common factors $W(q^{-1})$ in all polynomials (only "nearly" because of the approximate nature of the identification).

Pole-zero cancellations

This type of situation can be identified by checking if some poles and zeros of the model (approximately) cancel each other out.

We exemplify using Matlab function `pzmap`, which shows the poles and zeros of G in the generic model:

$$y(k) = G(q^{-1})u(k) + v(k)$$

For the ARMAX example, $G(q^{-1}) = \frac{W(q^{-1})B_0(q^{-1})}{W(q^{-1})A_0(q^{-1})}$, so the roots of W are both poles and zeros and (approximately) cancel each other out.

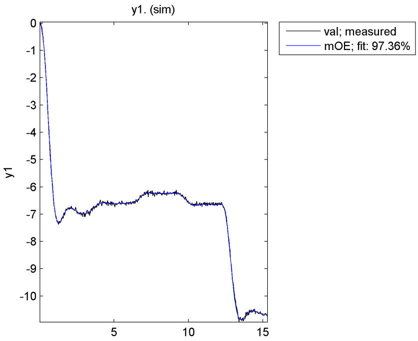
This idea extends to other model types besides ARMAX.

Matlab: overparameterized OE model

On the same data as for correlation tests (recall system has order $n = 3$):

```
mOE = oe(id, [5, 5, 1]);
```

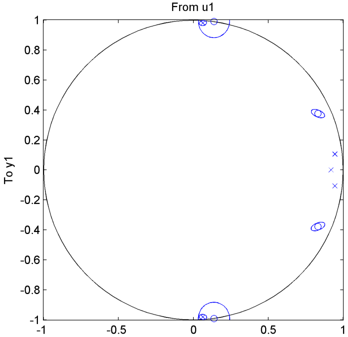
Looking at the validation data, the model is accurate:



Matlab: testing for pole-zero cancellations

```
pzmap(mOE, 'sd', nsd);
```

Arguments `'sd', nsd` specify the confidence region as a number of standard deviations. Here we take `nsd`=1.96 to get 0.95 confidence.



Two pairs of poles and zeros have overlapping confidence regions ⇒ likely they are canceling each other. This indicates the order should be 3 (the true system order).