

System Identification

Control Engineering EN, 3rd year B.Sc.
Technical University of Cluj-Napoca
Romania

Lecturer: Lucian Buşoniu



Part VII

Input signals

Motivation

Identification methods require inputs that satisfy certain conditions.

Examples:

- Transient analysis requires step or impulse inputs.
- Correlation analysis preferably works with white-noise input.
- PEM and IV require “sufficiently informative” inputs – the property was left informal in previous lectures.

Topic

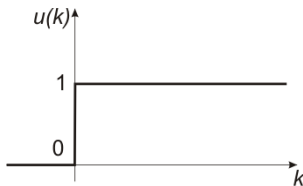
In this part:

- We **revisit** some types of input signals that were already used.
- A few **new types of input signals** are described.
- The old and new signals are **characterized** by their properties relevant for system identification.

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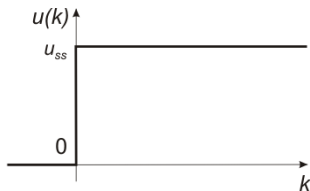
- 1 Common input signals
 - Step, impulse, sum of sines, white noise
 - Pseudo-random binary sequence
- 2 Signal properties
- 3 Characterization of common input signals

Step input



Left: Unit step:

$$u(k) = \begin{cases} 0 & k < 0 \\ 1 & k \geq 0 \end{cases}$$

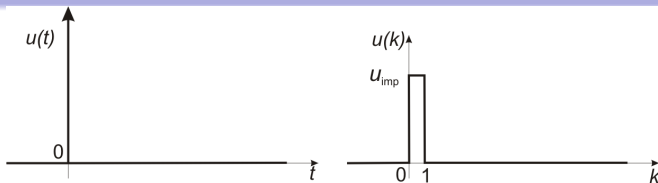


Right: Step of arbitrary magnitude:

$$u(k) = \begin{cases} 0 & k < 0 \\ u_{ss} & k \geq 0 \end{cases}$$

Remark: These are discrete-time reformulations of the continuous-time variants.

Impulse input



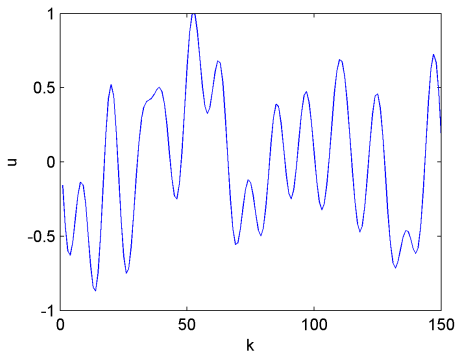
In discrete-time, we cannot freely approximate the ideal impulse (left), since the signal can only change values at the sampling instants.

Right: Discrete-time impulse realization:

$$u(k) = \begin{cases} u_{\text{imp}} & k = 0 \\ 0 & \text{otherwise} \end{cases}$$

- When $u_{\text{imp}} = \frac{1}{T_s}$, the integral of the signal is 1 and we get an approximation of the continuous-time impulse.
- When $u_{\text{imp}} = 1$ (e.g. in correlation analysis), we get a “unit” discrete-time impulse.

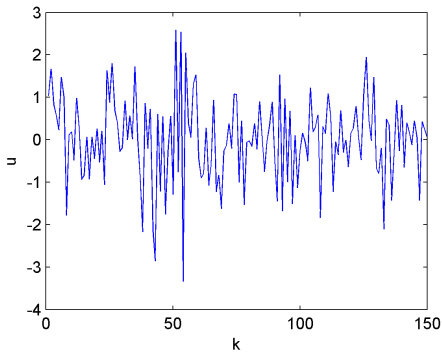
Sum of sines



$$u(k) = \sum_{j=1}^m a_j \sin(\omega_j k + \varphi_j)$$

- a_j : amplitudes of the m component sines
- ω_j : frequencies, $0 \leq \omega_1 < \omega_2 < \dots < \omega_m \leq \pi$
- φ_j : phases

White noise

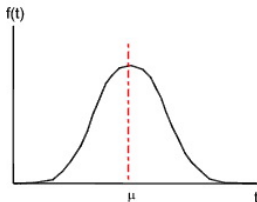


Recall zero-mean white noise: mean 0, different steps uncorrelated.

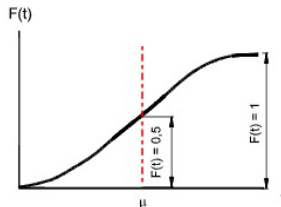
White noise (continued)

In the example figure, values were independently drawn from a zero-mean Gaussian distribution:

$$u(k) \sim \mathcal{N}(0, \sigma^2), \text{ with pdf } f(u) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{u^2}{2\sigma^2}\right)$$



Normal (Gaussian) pdf Distribution

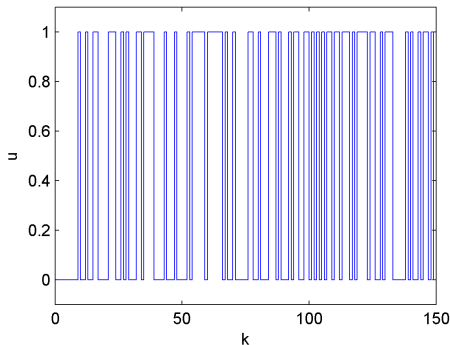


Cumulative Distribution Function

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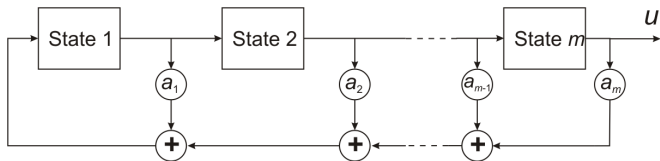
Pseudo-random binary sequence (PRBS)



A signal that switches between two discrete values, generated with a specific algorithm.

Interesting because it approximates white noise, and so it inherits some of the useful properties of white noise (formalized later).

PRBS generator



PRBS can be generated with a **linear shift feedback register** as in the figure. All signals and coefficients are binary.

At each discrete step k :

- State x_i transfers to state x_{i+1} .
- State x_1 is set to the modulo-two addition of states on the feedback path (if $a_i = 1$ then x_i is added, if $a_i = 0$ then it is not).
- Output $u(k)$ is collected at state x_m .

Modulo-two addition

Formula/truth table of modulo-two addition:

$$p \oplus q = \begin{cases} 0 & \text{if } p = 0, q = 0 \\ 1 & \text{if } p = 0, q = 1 \\ 1 & \text{if } p = 1, q = 0 \\ 0 & \text{if } p = 1, q = 1 \end{cases}$$

...also known as XOR (eXclusive OR)

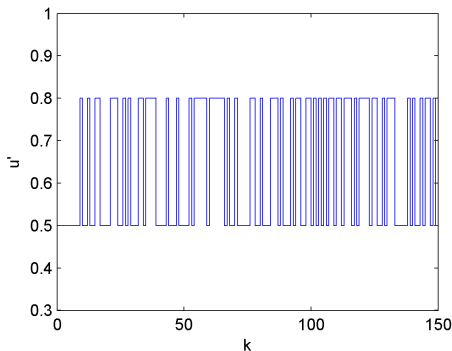
Remark: such a feedback register is easily implemented in hardware.

Arbitrary-valued PRBS

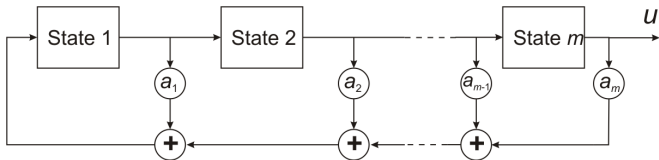
To obtain a signal $u'(k)$ taking values a, b instead of 0, 1, shift & scale the original signal $u(k)$:

$$u'(k) = a + (b - a)u(k)$$

Example for $a = 0.5$, $b = 0.8$:



State space representation



$$x_1(k+1) = a_1 x_1(k) \oplus a_2 x_2(k) \oplus \cdots \oplus a_m x_m(k)$$

$$x_2(k+1) = x_1(k)$$

$$\vdots$$

$$x_m(k+1) = x_{m-1}(k)$$

$$u(k) = x_m(k)$$

$x(k) = [x_1(k), \dots, x_m(k)]^\top$ compactly denotes the state vector of m variables

State space representation: matrix form

$$x(k+1) = \begin{bmatrix} a_1 & a_2 & \dots & a_{m-1} & a_m \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix} \otimes x(k) =: A \otimes x(k)$$

$$u(k) = [0 \ 0 \ \dots \ 0 \ 1]x(k) =: Cx(k)$$

where \otimes symbolically indicates that the additions in the matrix product are performed modulo 2.

Period of PRBS

- The PRBS algorithm is deterministic, so the current state $x(k)$ fully determines the future output
- ⇒ **Period** (number of steps until sequence repeats) at most 2^m
- The identically zero state is undesirable, as the future sequence would always remain 0
- ⇒ Maximum practical period is $P = 2^m - 1$

A PRBS with period $P = 2^m - 1$ is called **maximum-length PRBS**.

Such PRBS have interesting characteristics, so they are preferred in practice.

Maximum-length PRBS

The period is determined by the feedback coefficients a_i .

The following coefficients must be 1 to achieve maximum length (all others 0):

m	Max period $2^m - 1$	Coefficients equal to 1
3	7	a_1, a_3
4	15	a_1, a_4
5	31	a_2, a_5
6	63	a_1, a_6
7	127	a_1, a_7
8	255	a_1, a_2, a_7, a_8
9	511	a_4, a_9
10	1023	a_3, a_{10}

Other working combinations of coefficients exist, and coefficients for larger m can be found in the literature.

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Mean and covariance

Given a random signal $u(k)$, its mean and covariance are defined:

$$\begin{aligned}\mu &= \text{E} \{u(k)\} \\ r_u(\tau) &= \text{E} \{[u(k + \tau) - \mu][u(k) - \mu]\}\end{aligned}$$

Recall:

- Mean and variance of random variables
- Related covariance function $r_u(\tau)$ in correlation analysis

Mean and covariance: deterministic signal

When the signal is deterministic (e.g. PRBS), the mean and covariance are redefined as:

$$\mu = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N u(k)$$

$$r_u(\tau) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N [u(k + \tau) - \mu][u(k) - \mu]$$

Note: $\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N \cdot$ is the same as $\mathbb{E}\{\cdot\}$ for a (well-behaved) random signal.

Generalization to vector signals $u(k) \in \mathbb{R}^{nu}$: interpret the sums elementwise, replace $[u(k + \tau) - \mu][u(k) - \mu]$ by $[u(k + \tau) - \mu][u(k) - \mu]^T$, an $nu \times nu$ covariance matrix.

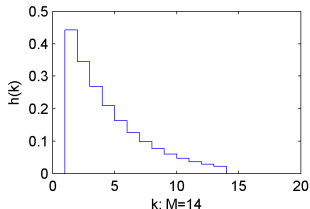
Persistent excitation

Next: Persistent excitation.

Motivation: We develop an *idealized* version of correlation analysis. This is only an intermediate motivating step, and the property is useful in many identification algorithms.

Finite impulse response (FIR) model:

$$y(k) = \sum_{j=0}^{M-1} h(j)u(k-j) + v(k)$$



Correlation analysis: Covariance estimation

Assuming $u(k)$, $y(k)$ are zero-mean, so the means do not need to be subtracted:

$$r_u(\tau) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N u(k + \tau)u(k)$$

$$r_{yu}(\tau) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N y(k + \tau)u(k)$$

In practice covariances must be estimated from finite datasets, but here we work with their ideal values (since this is only a motivating example, which we do not actually implement).

Correlation analysis: Identifying the FIR

Taking M equations to find the FIR parameters, we have:

$$\begin{bmatrix} r_{yu}(0) \\ r_{yu}(1) \\ \vdots \\ r_{yu}(M-1) \end{bmatrix} = \begin{bmatrix} r_u(0) & r_u(1) & \dots & r_u(M-1) \\ r_u(1) & r_u(0) & \dots & r_u(M-2) \\ \vdots & \vdots & \ddots & \vdots \\ r_u(M-1) & r_u(M-2) & \dots & r_u(0) \end{bmatrix} \cdot \begin{bmatrix} h(0) \\ h(1) \\ \vdots \\ h(M-1) \end{bmatrix}$$

Denote the matrix in the equation by $R_u(M)$, the **covariance matrix**.

Persistent excitation: formal definition

Definition

A signal $u(k)$ is **persistently exciting (PE) of order n** if $R_u(n)$ is positive definite.

A matrix $A \in \mathbb{R}^{n \times n}$ is positive definite if the scalar $h^T A h > 0$ for any nonzero vector $h \in \mathbb{R}^n$. Note that A must be nonsingular.

Examples:

- $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is positive definite. Denote $h = \begin{bmatrix} a \\ b \end{bmatrix}$, then
 $h^T A h = a^2 + b^2$.
- $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ is not positive definite. Counterexample: $h = \begin{bmatrix} a \\ -a \end{bmatrix}$,
 $h^T A h = -2a^2$.

PE in correlation analysis

If the order of PE is M , then $R_U(M)$ is positive definite, hence invertible and the linear system from correlation analysis can be solved to find an FIR of length M .

So an order M of PE means that an FIR model of length M is identifiable.

General role of PE

Beyond FIR, PE plays a role in all parametric system identification methods. Recall the requirements for the guarantees of prediction error and instrumental variable methods:

Assumptions

...

- 2 The input signal $u(k)$ is “sufficiently informative”.

...

These assumptions can be precisely stated in terms of a **large enough order of PE**. The required order is equal to (or a multiple of) the number of parameters n that must be estimated.

Covariance alternatives

One of the two following definitions can be used for the covariance:

$$r_u(\tau) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N u(k + \tau)u(k) \quad (7.1)$$

$$r_u(\tau) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N [u(k + \tau) - \mu][u(k) - \mu] \quad (7.2)$$

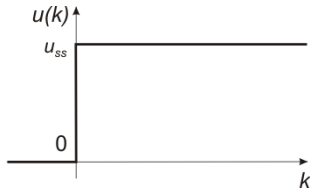
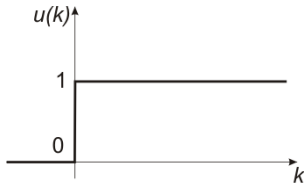
When $u(k)$ is not zero-mean, these two definitions lead to different orders of PE (larger by 1 for the first definition).

Although the first definition is not really the true covariance, it is convenient to compute, so we will use it when examining PE in the sequel.

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Step input



Take the more general, non-unit step:

$$u(k) = \begin{cases} 0 & k < 0 \\ u_{ss} & k \geq 0 \end{cases}$$

Step input: Mean and covariance

Mean and covariance:

$$m = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=0}^{N-1} u(k) = u_{ss}$$

$$r_u(\tau) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=0}^{N-1} u(k + \tau)u(k) = u_{ss}^2$$

Note the signal starts from $k = 0$, so the summation is modified (unimportant to the final result).

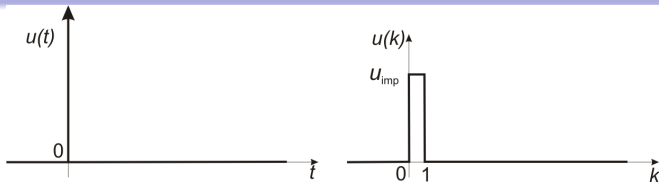
Step input: Order of PE

Covariance matrix:

$$R_U(n) = \begin{bmatrix} r_u(0) & r_u(1) & \dots & r_u(n-1) \\ r_u(1) & r_u(0) & \dots & r_u(n-2) \\ \vdots & & & \\ r_u(n-1) & r_u(n-2) & \dots & r_u(0) \end{bmatrix} = \begin{bmatrix} u_{ss}^2 & u_{ss}^2 & \dots & u_{ss}^2 \\ u_{ss}^2 & u_{ss}^2 & \dots & u_{ss}^2 \\ \vdots & & & \\ u_{ss}^2 & u_{ss}^2 & \dots & u_{ss}^2 \end{bmatrix}$$

This matrix has rank 1, so a step input is **PE of order 1**.

Impulse input



Recall discrete-time realization:

$$u(k) = \begin{cases} \frac{1}{T_s} & k = 0 \\ 0 & \text{otherwise} \end{cases}$$

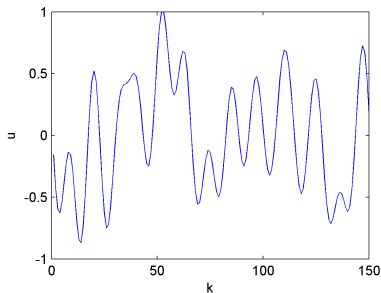
Mean and covariance:

$$m = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=0}^{N-1} u(k) = 0$$

$$r_u(\tau) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=0}^{N-1} u(k + \tau)u(k) = 0$$

⇒ The impulse is not PE of any order.

Sum of sines



$$u(k) = \sum_{j=1}^m a_j \sin(\omega_j k + \varphi_j), \quad 0 \leq \omega_1 < \omega_2 < \dots < \omega_n \leq \pi$$

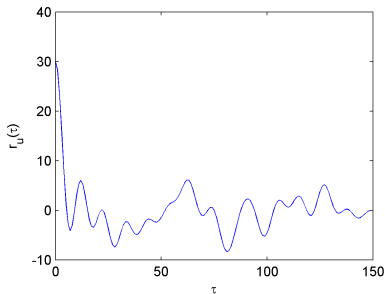
Mean and covariance:

$$m = \begin{cases} a_1 \sin(\varphi_1) & \text{if } \omega_1 = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$r_u(\tau) = \sum_{j=1}^{m-1} \frac{a_j^2}{2} \cos(\omega_j \tau) + \begin{cases} a_m^2 \sin^2 \varphi_m & \text{if } \omega_m = \pi \\ \frac{a_m^2}{2} \cos(\omega_m \tau) & \text{otherwise} \end{cases}$$

Sum of sines (continued)

For the multisine exemplified before, the covariance function is:



A multisine having m components is **PE of order n** with:

$$n = \begin{cases} 2m & \text{if } \omega_1 \neq 0, \omega_m \neq \pi \\ 2m - 1 & \text{if } \omega_1 = 0 \text{ or } \omega_m = \pi \\ 2m - 2 & \text{if } \omega_1 = 0 \text{ and } \omega_m = \pi \end{cases}$$

White noise: Mean and covariance

Take a zero-mean white noise signal of variance σ^2 , e.g. drawn from a Gaussian:

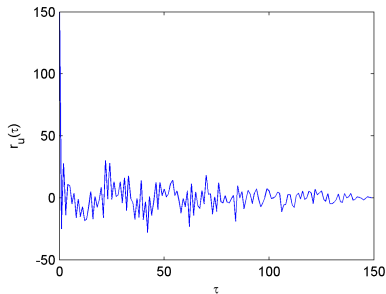
$$u(k) \sim \mathcal{N}(0, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right)$$

Then, by definition:

$$\mu = 0$$
$$r_u(\tau) = \begin{cases} \sigma^2 & \text{if } \tau = 0 \\ 0 & \text{otherwise} \end{cases}$$

White noise: Covariance example

Covariance function of white noise signal exemplified before:



White noise: Order of PE

Covariance matrix:

$$R_u(n) = \begin{bmatrix} r_u(0) & r_u(1) & \dots & r_u(n-1) \\ r_u(1) & r_u(0) & \dots & r_u(n-2) \\ \vdots & & & \\ r_u(n-1) & r_u(n-2) & \dots & r_u(0) \end{bmatrix}$$
$$= \begin{bmatrix} \sigma^2 & 0 & \dots & 0 \\ 0 & \sigma^2 & \dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots & \sigma^2 \end{bmatrix} = \sigma^2 I_n$$

where I_n = the identity matrix, positive definite.

⇒ for any n , $R_u(n)$ positive definite — white noise is **PE of any order**.

Question

Given the information above, why does correlation analysis prefer white noise to other input signals, in order to identify the FIR?

PRBS: Mean

Consider a 0, 1-valued, maximum-length PRBS with m bits:
 $P = 2^m - 1$, a large number.

Then its state $x(k)$ will contain all possible binary values with m digits except 0.

Signal $u(k)$ is the last position of $x(k)$, which takes value 1 a number of 2^{m-1} times, and value 0 a number of $2^{m-1} - 1$ times.

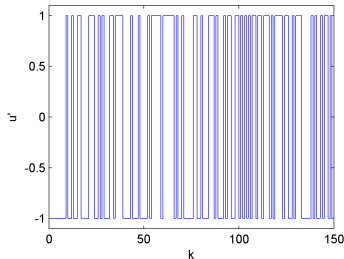
⇒ Mean value:

$$\mu = \frac{1}{P} \sum_{k=1}^P u(k) = \frac{1}{P} 2^{m-1} = \frac{(P+1)/2}{P} = \frac{1}{2} + \frac{1}{2P} \approx \frac{1}{2}$$

where the approximation holds for large P .

PRBS: Covariance

Consider a zero-mean PRBS, scaled between $-a$ and a :



$$u'(k) = -a + 2au(k)$$

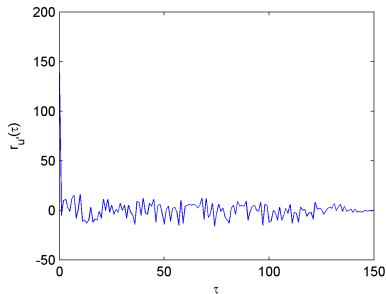
Then:

$$\mu = -a + 2a\left(\frac{1}{2} + \frac{1}{2P}\right) = \frac{a}{P} \approx 0$$

$$r_u(\tau) = \begin{cases} 1 - \frac{1}{P^2} \approx 1 & \text{if } \tau = 0 \\ -\frac{1}{P} - \frac{1}{P^2} \approx -\frac{1}{P} \approx 0 & \text{otherwise} \end{cases}$$

PRBS: Covariance example

Covariance function of the zero-mean PRBS above:



So, PRBS **behaves similarly to white noise** (similar covariance function). Combined with the ease of generating it, this property makes PRBS very useful in system identification.

PRBS: Order of PE

A maximum-length PRBS is **PE of exactly order P** , the period (and not larger).

Exercise

Using the approximate formula that ignores the terms $\frac{1}{P^2}$:

$$r_u(\tau) \approx \begin{cases} 1 & \text{if } \tau = 0 \\ -\frac{1}{P} & \text{otherwise} \end{cases}$$

take a small value of $P \geq 2$ and show that the PRBS is exactly of PE order P .

Hint: construct $R_u(n)$ for $n = P$ and show that it is rank P , then for $n > P$ and show it is *still* only of rank P . This can be done by showing that columns $P + 1, P + 2, \dots$ are linear combinations of the first P columns.