Optimistic Optimization

Lucian Bușoniu

20 May 2013
1. Problem & motivation
2. DOO: Deterministic optimistic optimization
3. SOO: Simultaneous optimistic optimization
4. Application: Multiagent consensus
These methods were published in:


1 is original reference, 2 is an extensive survey including applications to control
**Optimization problem**

\[
\max_{x \in X} f(x)
\]

**Assumption 1**

Function \( f : X \rightarrow \mathbb{R} \) is Lipschitz-continuous with respect to a semimetric \( \ell : X \times X \rightarrow \mathbb{R} \):

\[
|f(x) - f(x')| \leq \ell(x, x')
\]

**Definition (Semimetric)**

A function \( \ell : X \times X \rightarrow \mathbb{R} \) satisfying:

- \( \ell(x, x') \geq 0 \)
- \( \ell(x, x') = \ell(x', x) \)
- \( \ell(x, x') = 0 \) if and only if \( x = x' \)

Intuitively: a notion of distance
Motivation

No method with guaranteed performance for any function

Especially when the metric $\ell$ is unknown
DOO idea

- Explore the space $X$ iteratively
- Always expand **optimistic** set, with largest upper bound:
  \[ b(X_i) = f(x_i) + \delta(X_i), \quad \text{diam.} \quad \delta(X_i) = \sup_{x, x' \in X_i} \ell(x, x') \]
- Until $n$ expansions exhausted
Partitioning

- In general, a hierarchical partitioning rule must be defined
- Set $X_{0,1} = X$ at depth 0 split into $X_{1,1}, \ldots, X_{1,K}$ at depth 1
- Each set $X_{d,i}$ at depth $d$ split into $K$ subsets at depth $d + 1$
Assumption 2

The sets $X_{d,i}$ in the hierarchical partitioning must:

a) Shrink with the depth:
   \[ \delta(X_{d,i}) \leq \delta_d \] for any set $i$ at $d$; $\delta_d$ decreases with $d$

b) Be well-shaped:
   each $\delta(X_{d,i})$ contains a ball in the semimetric $\ell$ having radius proportional to $\delta_d$, $B(x_{d,i}, \nu \delta_d)$
DOO algorithm

initialize tree with root $X_{0,1} = X$
for $t = 1$ to $n$ do
    $X_{d,i}^\dagger \leftarrow \arg\max_{X_{d,i} \in \text{leaves}} b(X_{d,i})$
    expand $X_{d,i}^\dagger$ (partition the set), adding children to tree
end for
output best sample $\hat{x}^* = \arg\max_{x_{d,i} \in \text{tree}} f(x_{d,i})$

(Munos, 2011)
Examples

- Quadratic function
- Rosenbrock banana function
An easy near-optimality guarantee

Denote the expanded set at iteration $t$ by $X_t^+$

- $b(X_t^+) \geq f^*$, otherwise it wouldn’t have been selected
- $f(x_t^+) \leq f^*$ by definition
- $f(\hat{x}^*) \geq f(x_t^+)$ because $\hat{x}^*$ maximizes $f$ on the tree
- So $f^* - f(\hat{x}^*) \leq \delta(X_t^+)$ at any $t$, and therefore $\leq \delta d^*$, where $d^*$ the deepest expanded depth
Near-optimality dimension

Definition (Near-optimality dimension)

Smallest $\beta$ so that the near-optimal sets:

$$X_\varepsilon = \{ x \in X \mid f^* - f(x) \leq \varepsilon \}$$

can be covered by (on the order of) $\varepsilon^{-\beta}$ balls of radius $\varepsilon$ in the semimetric $\ell$

$\beta$ measures how closely $\ell$ captures the smoothness of $f$
Consider a partition with exponentially decreasing sets, $\delta_d = \gamma^d$, $\gamma < 1$. Then the solution returned by DOO satisfies:

$$f^* - f(\hat{x}^*) \approx \begin{cases} 
    n^{-1/\beta} & \text{if } \beta > 0 \\
    \gamma^{cn} & \text{if } \beta = 0
\end{cases}$$
Example: zero dimension

- Take $f(x^*) - f(x) \approx |x^* - x|$ and $\ell(x, x') = |x - x'|$
- $X_\varepsilon$ = an interval of length $\varepsilon$, which is also an $\ell$-ball of size $\varepsilon$
- So it takes a constant = $\varepsilon^0$ number of balls to cover $X_\varepsilon$, and $\beta = 0$

(taken from Munos)
Example: positive dimension

- If $f(x^*) - f(x) \approx |x^* - x|^2$, $X_\varepsilon$ is an interval of length $\sqrt{\varepsilon}$.
- When $\ell(x, x') = |x - x'|^2$, a $\ell$-ball of size $\varepsilon$ is also an interval of length $\sqrt{\varepsilon}$, and $\beta = 0$.
- When $\ell(x, x') = |x - x'|$, a $\ell$-ball of size $\varepsilon$ is an interval of length $\varepsilon$, so it takes $\varepsilon / \sqrt{\varepsilon} = \varepsilon^{-1/2}$ balls to cover $X_\varepsilon$, and $\beta = 1/2$.

(taken from Munos)
Example: semimetric mismatch

Influence of semimetric (mis)match for a quadratic function
1. Problem & motivation

2. DOO: Deterministic optimistic optimization

3. SOO: Simultaneous optimistic optimization

4. Application: Multiagent consensus
What if $\ell / \delta$ unknown? (i.e., smoothness of $f$ unknown)

Expand **all potentially optimistic sets** $X_{d,i}$, for which:

$$f(x_{d,i}) \geq f(x_{d',j})$$

for all leaves $j$ at smaller depths $d' \leq d$
**SOO algorithm**

initialize tree with root $X_{0,1} = X$

**repeat** at each iteration $t = 1, 2, \ldots$

**for** $d = 0, \ldots, \min\{\text{current tree depth}, d_{\text{max}}(t)\}$ **do**

- $X_{d,i}^{\dagger} \leftarrow \arg \max_{X_{d,i} \in \text{leaves at } d} f(x_{d,i})$

- **if** $f(x_{d,i}^{\dagger}) \geq f(x_{d',j})$ $\forall$ leaves $j$ at $d' \leq d$ **then**
  - expand $X_{d,i}^{\dagger}$

**end if**

**end for**

**until** $n$ expansions performed

**output** best sample $\hat{x}^* = \arg \max_{x_{d,i} \in \text{tree}} f(x_{d,i})$

(Munos, 2011)
Examples

- Quadratic function
- Rosenbrock banana function
Theorem

Consider a partition with exponentially decreasing sets, \( \delta_d = \gamma^d, \gamma < 1 \). Take \( d_{\text{max}}(t) = \sqrt{t} \), then the solution returned by SOO satisfies:

\[
 f^* - f(\hat{x}^*) \approx \begin{cases} 
 n^{-\frac{1}{2\beta}} & \text{if } \beta > 0 \\
 \gamma c' n & \text{if } \beta = 0 
\end{cases}
\]
1. Problem & motivation

2. DOO: Deterministic optimistic optimization

3. SOO: Simultaneous optimistic optimization

4. Application: Multiagent consensus
Consensus in nonlinear multiagent systems

- Agents with nonlinear dynamics $x_{i,k+1} = f_i(x_{i,k}, u_{i,k})$
- **Consensus problem**: agents must reach agreement on (some) state variables
- Communication on an incomplete graph
- **Challenge**: No solution for general $f$
OO for consensus

1. Design target states with a classical consensus method
2. Use DOO or SOO to optimize action sequences in order to reach within $\varepsilon$ of target states

- **Consensus guaranteed** under conditions on $f$
- Tradeoff: length of action sequence must be known and small
Consensus of multiple robot arms