Discounted optimal control with continuous actions using optimistic planning

Lucian Buşoniu, Előd Páll, Rémi Munos

TUCluj, Romania, and Google DeepMind, UK
Contact: lucian@busoniu.net

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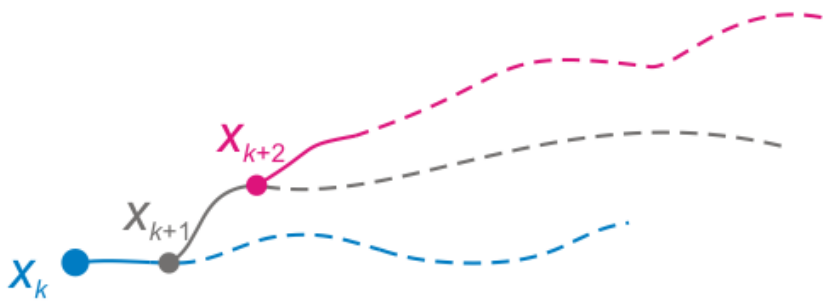
Setting

- Deterministic system $x_{k+1} = f(x_k, u_k)$
- Reward function $\rho(x_k, u_k)$
- Locally at state x_0, find action sequence $\boldsymbol{u}_\infty = (u_0, u_1, \dots)$ to maximize discounted value:

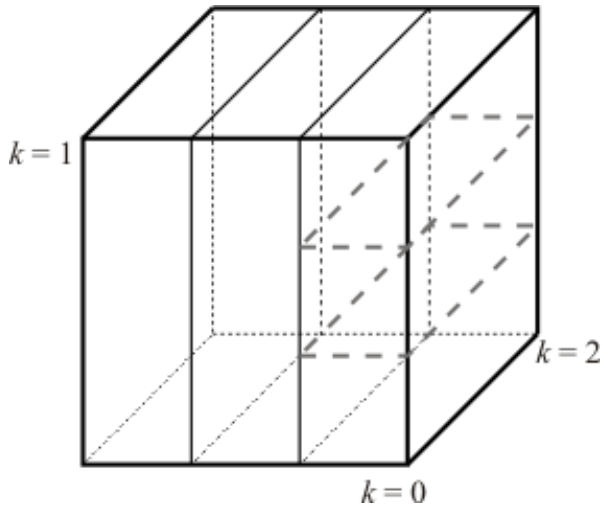
$$v(\boldsymbol{u}_\infty) = \sum_{k=0}^{\infty} \gamma^k \rho(x_k, u_k)$$

  where **discount factor** $\gamma \in [0, 1)$
- Find near-optimal sequence, repeat in receding horizon



Optimistic planning (OP): Main idea

initialize **set of all possible sequences**
repeat
    select most promising, **optimistic** set
    refine selected set
until computation budget n exhausted
return sequence in best set



Bandit-based optimization; branch & bound if deterministic

Advantages of OP

- **Near-optimality guarantees** as a function of computation n and of complexity m of the problem:

$$\text{error} = O(g(n, m))$$

(Munos, 2014)

- ...for general nonlinear dynamics and rewards

Assumptions

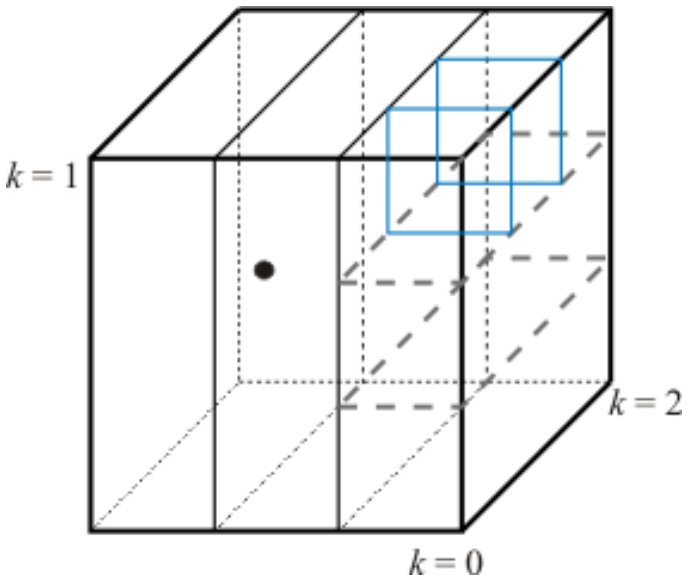
- Rewards $r \in [0, 1]$
- Action space $U = [0, 1]$
  (can be extended to compact multidimensional U)
- Lipschitz dynamics and rewards:

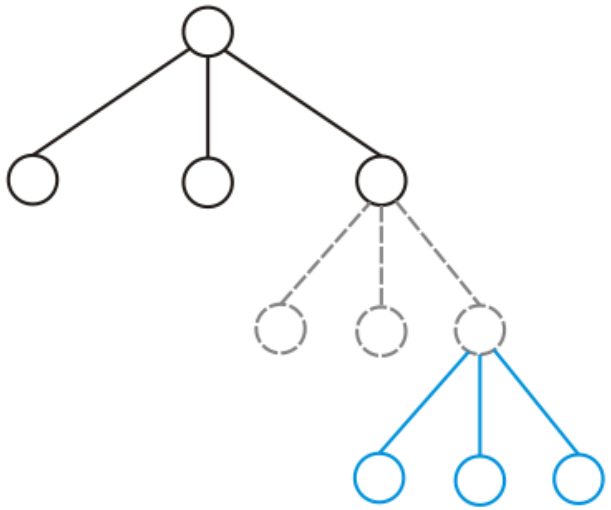
$$\left\| f(x, u) - f(x', u') \right\| \leq L_f \left(\left\| x - x' \right\| + \left| u - u' \right| \right)$$
$$\left| \rho(x, u) - \rho(x', u') \right| \leq L_\rho \left(\left\| x - x' \right\| + \left| u - u' \right| \right)$$

- $\gamma L_f < 1$: most restrictive

Search refinement

- Split U^∞ iteratively, leading to a tree of hyperboxes





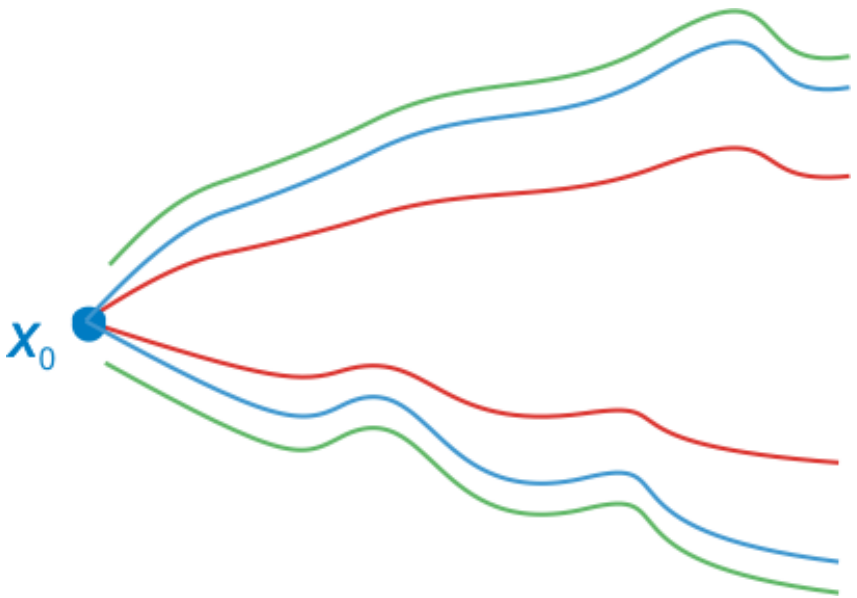
- Each box i only represents explicitly dimensions already split, $k = 0, \ldots, K_i - 1$
- Box i has value $v(i) = \sum_{k=0}^{K_i-1} \gamma^k r_{i,k+1}$, rewards of center sequence

Lipschitz value function

- For any two action sequences $\boldsymbol{u}_\infty, \boldsymbol{u}'_\infty$:

$$\left|v(\boldsymbol{u}_\infty) - v(\boldsymbol{u}'_\infty)\right| \leq \frac{L_\rho}{1-\gamma L_f} \sum_{k=0}^{\infty} \gamma^k \left|u_k - u'_k\right|$$

- Intuition: states (and so rewards) may diverge somewhat, but divergence controlled due to $\gamma L_f < 1$

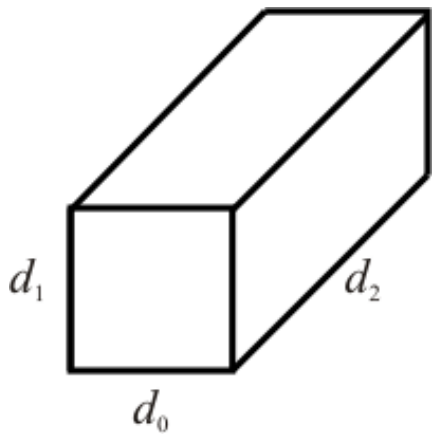


Box upper bound

- For any sequence $\boldsymbol{u}_\infty$ in box i:

$$v(\boldsymbol{u}_\infty) \leq v(i) + \frac{\max\{1, L_\rho\}}{1 - \gamma L_f} \sum_{k=0}^{\infty} \gamma^k d_{i,k} := b(i)$$

- $d_{i,k}$ length of dimension k, 1 if not split yet



- $b(i)$ **b-value** of box i

Diameter and dimension selection

- **Diameter** $\delta(i) := \frac{\max\{1, L_\rho\}}{1-\gamma L_f} \sum_{k=0}^{\infty} \gamma^k d_{i,k}$
  $=$ uncertainty on values in the box
- **Impact** of dimension k on uncertainty is $\gamma^k d_{i,k}$

$\Rightarrow$ when splitting a box, choose dimension with largest impact, to reduce uncertainty the most

- Always split into odd $M > 1/\gamma$ pieces

OPC algorithm

initialize tree with root box U^∞
while n not exhausted **do**
    select **optimistic** leaf box $i^\dagger = \arg\max_{i \in \mathcal{L}} b(i)$
    select **max-impact** dimension $k^\dagger = \arg\max_k \gamma^k d_{i^\dagger,k}$
    split $i^\dagger$ along $k^\dagger$, creating M children on the tree
end while
return best center sequence seen, $i^* = \arg\max_i v(i)$

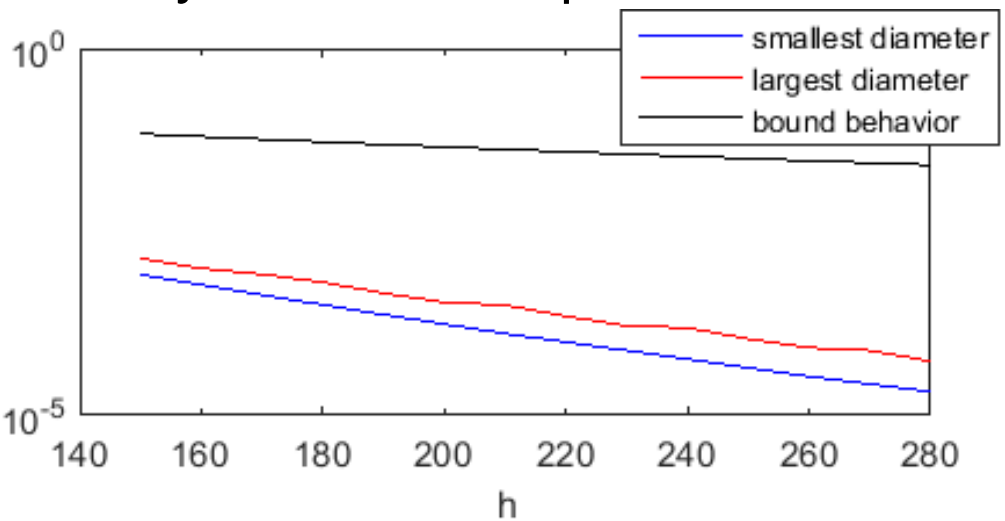
Diameter bound

Lemma

Given depth in tree h =total number of splits:

$$\delta(i) = \tilde{O}(\gamma^{\sqrt{2h\frac{\tau-1}{\tau^2}}}), \text{ where } \tau = \left\lceil \frac{\log 1/M}{\log \gamma} \right\rceil$$

Diameters vary by the order of splits, but they all converge to 0 roughly exponentially in $\sqrt{h}$. Example:



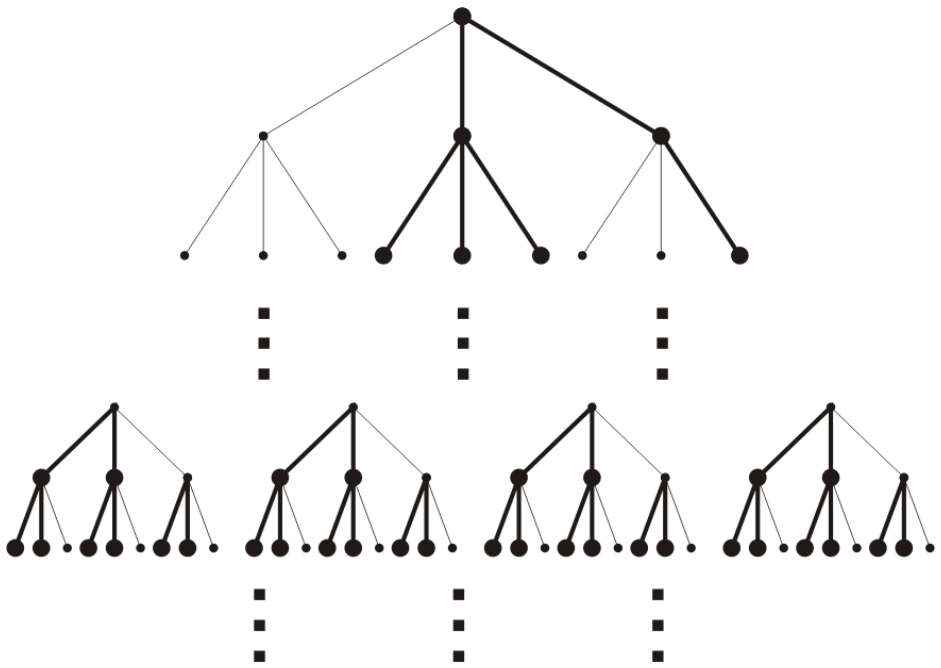
Complexity measure

- OPC only expands in near-optimal subtree:

$$\mathcal{T}^* = \{i \in \mathcal{T} \mid v^* - v(i) \leq \delta(i)\}$$

  (nodes that cannot be eliminated as suboptimal)
- Define $m \in [1, M]$ = asymptotic branching factor of $\mathcal{T}^*$: **problem complexity measure**

E.g. $m = 2$, $M = 3$:



Performance guarantee

Theorem

After spending n model calls, OPC suboptimality is:

$$v^* - v(i^*) = \begin{cases} \tilde{O}(\gamma^{\sqrt{\frac{2(\tau-1)\log n}{\tau^2 \log m}}}), & \text{if } m > 1 \\ \tilde{O}(\gamma^{n^{1/4}} b), & \text{if } m = 1 \end{cases}$$

- Convergence rate modulated by problem complexity m, faster when m smaller
- When $m = 1$, convergence is fast, with power $n^{1/4}$
- When $m > 1$, we pay for generality: exponential computation m^h to reach depth h

Related planning algorithms

- Inspired by optimistic optimization (DOO & SOO)

(Munos 2011)

- HOLOP, HOOT: fixed finite horizon

(Weinstein et al. 2012, Mansley et al. 2011)

- Lipschitz planning (LP): different dimension selection, no guarantees

(Hren 2012)

- SOOP: no Lipschitz constants, no guarantees

(ADPRL 2013)

Example: Quanser pendulum

System:

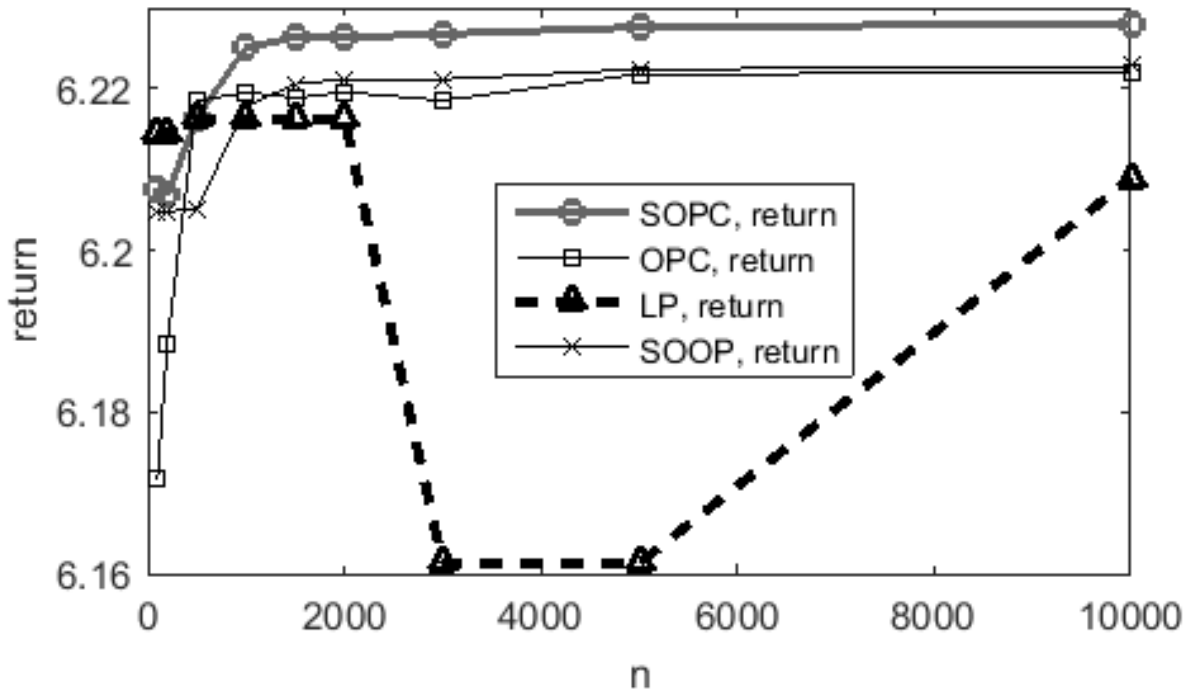
- x = rod angle α, base angle θ, angular velocities
- u = motor voltage $\in [-9, 9]$ V
- Sampling time $T_s = 0.05$

Goal: stabilize pointing up:

- $\rho = -\alpha^2 - \theta^2 - .005(\dot{\alpha}^2 + \dot{\theta}^2) - .05u^2$, normalized to $[0, 1]$
- Discount factor $\gamma = 0.85$
- Swingup required

OPC versus other planners

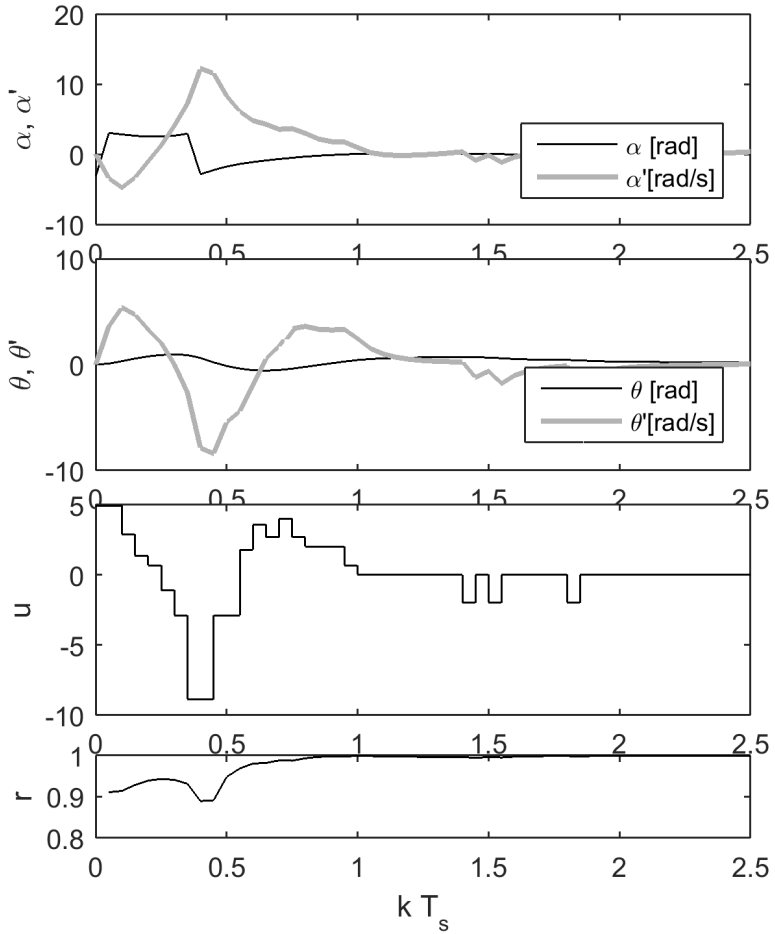
OPC with $L_f = L_\rho = 1.1$, tighter diameter formula
Parameters of algorithms optimized



$\Rightarrow$ OPC theoretically nice,
beaten by **simultaneous** algorithms in practice

Controlled trajectory (SOPC)

$n = 5000$ model calls; note adaptive discretization of control magnitude



Real-time control (SOPC)

Conclusions and next steps

Optimistic planning with continuous actions

- General nonlinear systems and cost functions
- Guaranteed near-optimality and convergence rate
- ... but beaten by simultaneous OPC in practice

$\Rightarrow$ Analyze SOPC

$+$ Eliminate assumption $\gamma L_f < 1$ using stability

Thank you!